Structure Theorem for Semisimple Rings: Wedderburn-Artin

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This document is a reorganization of some material from [1], with a view towards forging a direct route to the Wedderburn Artin theorem. Let R be a ring, which will always mean ring-with-1.

1 Background

1.1 Semisimple Modules

A left R-module M is *simple* if it is nontrivial and has no proper nontrivial submodules. A left R-module M is *semisimple* in case it is generated by its simple submodules.

Theorem 1: If $_RM$ is semisimple, then it is a direct sum of some of its simple submodules.
Proof: Let $\mathscr{T}$ be the set of simple submodules of M. A set of submodules is said to be *independent* if each submodule trivially intersects the span of the others. Let $\mathscr{T}' \subseteq \mathscr{T}$ be a maximal independent subset of $\mathscr{T}$ (use Zorn's lemma). We need only show that $M = \sum \mathscr{T}'$. Suppose otherwise; that is, suppose that $M \setminus \sum \mathscr{T}' = \sum \mathscr{T} \setminus \sum \mathscr{T}'$ is nonempty. There is some $T \in \mathscr{T}$ that is not contained in $\sum \mathscr{T}'$, and hence (by simplicity) intersects it trivially. Then $\mathscr{T}' \cup \{T\}$ is independent, contradicting the maximality of $\mathscr{T}'$.

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Theorem 2: A submodule N of a semisimple module M is a direct summand. Further, if M is the direct sum of simple submodules $\bigoplus_{\alpha\in A} T_\alpha$ then N is *isomorphic* to $\bigoplus_{\alpha\in A'} T_\alpha$ for some $A' \subseteq A$.
Proof: Let $A'' \subseteq A$ be a maximal subset with respect to the property that $\{T_\alpha \upharpoonright \alpha \in A''\} \cup \{N\}$ is independent. We must have $N + \sum_{A''} T_\alpha = M$, for otherwise there is some T_α, with $\alpha \in A \setminus A''$, which is not contained in $N + \sum_{A''} T_\alpha$ and therefore intersects it trivially (this contradicts the maximality of A''). Therefore we have

$$M = N \ \oplus \ \bigoplus_{\alpha\in A''} T_\alpha \ .$$

Let $A' = A \setminus A''$. It is easy to see that

$$N \cong \bigoplus_{\alpha\in A'} T_\alpha$$

since they are both direct-sum-complements to $\bigoplus_{A''} T_\alpha$.

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Theorem 3: Let $(T_\alpha)_{\alpha\in A}$, $(T_\beta)_{\beta\in B}$ be families of simple submodules of ${}_RM$, and suppose that

$$\sum_{\alpha\in A} T_\alpha \;\cap\; \sum_{\beta\in B} T_\beta \;\neq\; 0$$

Then $T_\alpha \cong T_\beta$ for some $\alpha\in A,\ \beta\in B$.
Proof: Let I denote the nontrivial intersection above. Applying Zorn's lemma as in the proof of (1), there are nonempty $A' \subseteq A$ and $B' \subseteq B$ such that

$$\sum_{\alpha\in A} T_\alpha = \bigoplus_{\alpha\in A'} T_\alpha\ , \qquad \sum_{\beta\in B} T_\beta = \bigoplus_{\beta\in B'} T_\beta\ .$$

Applying (2) to I, there are nonempty $A'' \subseteq A'$ and $B'' \subseteq B'$ such that

$$I \cong \bigoplus_{\alpha\in A''} T_\alpha \cong \bigoplus_{\beta\in B''} T_\beta\ .$$

Choose some $\alpha \in A''$ and consider the image $\overline{T}_\alpha$ of T_α in $\bigoplus_{B''} T_\beta$ under the above isomorphism. Apply (2) to $\overline{T}_\alpha$ to see that it is isomorphic to some T_β.

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1.2 Traces and Socles

If $\mathscr{U}$ is a class of R-modules, and if ${}_RM$ is a left R-module, then

$$\mathrm{Tr}_M(\mathcal{U}) := \sum\{\mathrm{im}\ h \,\rfloor\, U \in \mathscr{U}\ \centerdot\ h : {}_RU \to {}_RM\}\ .$$

It is the largest submodule of ${}_RM$ generated by $\mathscr{U}$.

The *socle* of ${}_RM$ is

$$\mathrm{Soc}({}_RM)\ :=\ \mathrm{Tr}_M(\text{ the class of simple left } R\text{-modules })\ .$$

It is the unique largest semisimple submodule of ${}_RM$.

A *homogeneous component* of $\mathrm{Soc}({}_RM)$ is $\mathrm{Tr}_M(T)$ for a simple ${}_RT$.

Theorem 4: Let M be a left R-module. Then Soc(M) is the direct sum of its homogeneous components.
Proof: Let $\mathscr{T}$ be a set of unique representatives of isomorphism classes of simple left R-modules. First observe that $\mathrm{Soc}(M)$ is spanned by its homogeneous components:

$$\begin{aligned}\mathrm{Soc}(M) &= \mathrm{Tr}_M(\mathscr{T})\\ &= \mathrm{Tr}_M(\bigoplus_{T\in\mathscr{T}} T)\\ &= \sum_{T\in\mathscr{T}} \mathrm{Tr}_M(T)\end{aligned}$$

To see that the sum is direct, we assume that

$$\mathrm{Tr}_M(T) \cap \sum_{\alpha\in A} \mathrm{Tr}_M(T_\alpha)\ \neq\ 0$$

for some simple left R-modules T and $(T_\alpha)_{\alpha\in A}$. The objective is then to show that $T \cong T_\alpha$ for some $\alpha \in A$. The trace in M of a simple module is the sum of its epimorphic images in M, each of which is necessarily isomorphic to the simple module (excluding trivial images). The intersection above can then be written as

$$\sum_{\beta\in B} T_\beta \;\cap\; \sum_{\gamma\in C} T_\gamma \;\neq\; 0$$

for families of simple submodules $(T_\beta)_{\beta\in B}$ and $(T_\gamma)_{\gamma\in C}$, where each T_β is isomorphic to T and each T_γ is isomorphic to T_α for some $\alpha \in A$. Applying (3) then completes the proof.

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Theorem 5: Traces in ${}_RR$ are not only submodules but also *two-sided* ideals.
Proof: Let $\mathscr{U}$ be a class of left R-modules. For any $r \in R$, $U \in \mathscr{U}$, and $h : {}_RU \to {}_RR$, we have a map

$${}_RU \xrightarrow{h} {}_RR \xrightarrow{\rho_r} {}_RR \ ,$$

since the right multiplication map ρ_r is a left R-homomorphism. It easily follows that $\mathrm{Tr}_{{}_RR}(\mathscr{U})$ is a two-sided ideal.

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1.3 Semisimple Rings

A ring R is said to be *semisimple* if ${}_RR$ is semisimple.

Theorem 6: Let R be semisimple. Every simple left R-module is isomorphic to a minimal left ideal in R.
Proof: Let ${}_RT$ be simple. Choose a nonzero $x \in T$, and define $\phi : {}_RR \to {}_RT$ by $r \mapsto rx$. This is clearly an epimorphism of left R-modules, and its kernel $\mathscr{M}$ is a maximal left ideal of R. By (2), $\mathscr{M}$ is a direct summand of ${}_RR$. It's direct sum complement is submodule of ${}_RR$ isomorphic to ${}^R/_{\mathscr{M}} \cong {}_RT$. This is the desired minimal left ideal.

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Theorem 7: Suppose ${}_RR = {}_RR_1 \oplus \cdots \oplus {}_RR_m$ internally, and suppose that each $R_i \subseteq R$ is a nonzero two-sided ideal. Then each R_i is a *ring* (i.e. has identity) and we obtain product decomposition of R as a ring:

$$R = R_1 \times \cdots \times R_m$$

Proof: Let $p_1, \cdots, p_m$ be the projection maps of the given left R-module direct sum decomposition. Note that a priori we only know that $p_i : {}_RR \to {}_RR_i$ is a left R-homomorphism. For each $1 \le i \le m$ define $e_i = p_i(1)$. Observe that

$$\begin{aligned} &e_1 + \cdots + e_m = 1 \qquad \text{and} \\ &e_i r e_j = 0 \quad \text{ for } i \neq j \text{ and any } r \in R. \end{aligned}$$

The first is a basic property of projections and the second follows from $e_i r e_j \in R_i \cap R_j = 0$ (where we've used the fact that each R_i is also a *right* ideal). From these properties we can show that the e_i are central; for any $r \in R$ we have

$$\begin{aligned} e_i r &= e_i r(e_1 + \cdots + e_m) \\ &= e_i r e_i \\ &= (e_1 + \cdots + e_m) r e_i = \ r e_i \ . \end{aligned}$$

Each e_i is a right identity for R_i:

$$p_i(r)e_i = p_i(r)p_i(1) = p_i(p_i(r)1) = p_i^2(r) = p_i(r) \quad \text{ for } r \in R.$$

It then follows from centrality of the e_i that they are also *left* identities for the respective R_i. That is, the R_i are rings. It also follows from centrality that the projections are *ring* homomorphisms. To see this, note that $p_i(r) = p_i(r1) = rp_i(1) = re_i$ for any $r \in R$. Then:

$$p_i(rs) = p_i^2(rs) = rse_i^2 = re_i se_i = p_i(r)p_i(s) \quad \text{ for } r, s \in R.$$

Finally, it is easy to check that the projection maps satisfy the necessary universal property for the alleged product decomposition of R as a ring.

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Theorem 8: Suppose ${}_RR$ has a direct sum decomposition $\bigoplus_{\alpha\in A} M_\alpha$. Then all but finitely many summands are trivial.
Proof: For each $\alpha \in A$ let $p_\alpha : {}_RR \to {}_RM_\alpha$ be the corresponding projection map. If, for a particular α, we have $p_\alpha(1) = 0$, then $M_\alpha = \text{im}\,(p_\alpha) = 0$. Of course $p_\alpha(1)$ can only be nonzero for finitely many $\alpha \in A$.

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Theorem 9: A ring with a simple left generator is simple.
Proof: Let S be a ring with simple left generator ${}_ST$. Then ${}_SS$ is a homomorphic image of a direct sum of copies of ${}_ST$. It follows from (2) that ${}_SS$ is *isomorphic* to a direct sum of copies of ${}_ST$, and according to (8) that direct sum is finite. Write the direct sum internally as

$${}_SS = \bigoplus_{i=1}^{n} T_i$$

for left ideals T_i of S each isomorphic to ${}_ST$. Let $I \subseteq S$ be a nonzero two-sided ideal of S. According to (2), there is a left ideal $T' \subseteq I$ of S such that $T' \cong T$. Furthermore it is a direct summand of ${}_SS$, so we have a projection map $p : {}_SS \to {}_ST'$. We will show that the two-sided ideal generated by T' is all of S by showing that it must contain each T_i. Consider any one of the T_i. Let $\phi : T' \xrightarrow{\cong} T_i$ be a choice of isomorphism, and define $e = \phi(p(1))$. Consider any $x \in T'$, say $x = p(y)$ with $y \in S$. We have

$$xe = x\phi(p(1)) = \phi(p(x)) = \phi(p^2(y)) = \phi(p(y)) = \phi(x)$$

It follows that $T'e$ contains T_i, and therefore that the two-sided ideal generated by T' contains T_i.

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2 Main Theorem

Theorem 10: Let R be a semisimple ring. Then there is a finite set $\mathscr{S} = \{T_1, \cdots, T_m\}$ of minimal left ideals of R such that:

1. $\mathscr{S}$ contains a unique representative of each isomorphism class of simple left R-module.
2. For each $T \in \mathscr{S}$, the T-homogeneous component of R is given by

   $$\text{Tr}_R(T) = RTR\ ,$$

   and it is a simple-artinian ring.
3. For each $T \in \mathscr{S}$, the T-homogeneous component of R is a matrix ring over a division ring:

   $$RTR \ \cong\ \mathbb{M}_n(D)\ ,$$

   where n is the composition length of RTR and $D = \text{End}({}_RT)$.
4. R is the "ring direct sum" (actually product)

   $$R \ =\ RT_1R \ \times\ \cdots\ \times\ RT_mR\ .$$

   That is, semisimple rings are products of matrix rings over division rings.

Proof: Since ${}_R R$ is semisimple, it is the direct sum of its homogeneous components (4). The homogeneous components of ${}_R R$ are the traces in ${}_R R$ of simple left R-modules. Every simple left R-module ${}_R T$ is isomorphic to a minimal left ideal of R (6), and in addition each such ${}_R T$ has nontrivial trace in ${}_R R$. Let $\mathscr{S}$ be a set consisting of a choice of minimal left ideal of R corresponding to each isomorphism class of simple left R-module. Since ${}_R R$ is the internal direct sum of the $\mathrm{Tr}_R(T)$ for $T \in \mathscr{S}$, we know that $\mathscr{S}$ is finite (8):

$$\mathscr{S} = \{T_1, \cdots, T_m\}$$

Each $\mathrm{Tr}_R(T_i)$ is a nonzero two-sided ideal (5), and we have a left R-module direct sum ${}_R R = \mathrm{Tr}_R(T_1) \oplus \cdots \oplus \mathrm{Tr}_R(T_m)$. It follows (7) that each $\mathrm{Tr}_R(T_i)$ is in fact a *ring*, and that we have a *ring direct sum*

$$R = \mathrm{Tr}_R(T_1) \times \cdots \times \mathrm{Tr}_R(T_m) \ .$$

Fix a $T \in \mathscr{S}$ and let S be the ring $\mathrm{Tr}_R(T)$. We have $T \subseteq S$, so T is a simple left ideal of S (simplicity is easy to see when one considers the ring direct sum decomposition). Since ${}_R T$ is a simple left generator of ${}_R S$, we have ${}_R S \cong {}_R T^{(A)}$ for some index set A. Viewing the direct sum as internal makes it clear that this decomposition is also one of S-modules: ${}_S S \cong {}_S T^{(A)}$. By (8), the direct sum is finite:

$${}_S S \cong {}_S T^{(n)} \ .$$

(Note that n is then the composition length of ${}_S S$). This also shows that ${}_S S$ has a simple left generator and is therefore a simple ring (9). It follows that S is a *minimal* two-sided ideal of R, and we may therefore write it as

$$S := \mathrm{Tr}_R(T) = RTR \ .$$

It remains only to prove (3). Defining $D = \mathrm{End}({}_S T)$ (a division ring by Schur's lemma), we have an isomorphism of rings:

right multiplication in S / evaluation at 1 / right matrix action on row vectors

$$S \cong \mathrm{End}({}_S S) \cong \mathrm{End}({}_S T^{(n)}) \cong \mathbb{M}_n(\mathrm{End}({}_S T)) = \mathbb{M}_n(D)$$

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References

[1] Anderson, F. and Fuller, K. [74]: Rings and Categories of Modules. New York-Heidelberg-Berlin: Springer-Verlag 1974.