The Prime Spectrum and Representation Theory of Generalized Weyl Algebras, with Applications to Quantized Algebras

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Abstract

This work develops the theory of generalized Weyl algebras (GWAs) in order to study generic quantized algebras. The ideas behind the classification of simple modules over GWAs are used to describe the noncommutative prime spectrum for certain GWAs. The primary example studied is the 2×2 reflection equation algebra $\mathcal{A} = \mathcal{A}_q(\mathrm{M}_2)$ in the case that q is not a root of unity, where the R-matrix used to define $\mathcal{A}$ is the standard one of type A. Simple finite dimensional $\mathcal{A}$-modules are classified, finite dimensional weight modules are shown to be semisimple, $\mathrm{Aut}(\mathcal{A})$ is computed, and the prime spectrum of $\mathcal{A}$ is computed along with its Zariski topology. It is shown that $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence and that it satisfies a Duflo-type theorem to some extent. The notion of a Poisson GWA is developed and used to explore the semiclassical limit of $\mathcal{A}$. For some other quantized algebras and their classical counterparts, the GWA theory is demonstrated as a means to study the prime spectrum.

1 Introduction

Generalized Weyl algebras, henceforth known as GWAs, form a class of noncommutative rings that was introduced by Bavula in [3]. Examples include the ordinary Weyl algebra and the classical and quantized universal enveloping algebras of $\mathfrak{sl}_2$. We shall focus on aspects of GWAs that provide tools for working with various quantized algebras. Quantized algebras are always constructed in terms of parameters, and their study typically splits into two realms depending on the parameters. When the parameters satisfy certain algebraic relations (which often amount to the parameters being roots of unity) the structure and representation theory of a quantized algebra take on a vastly different character than it would in the "generic parameters" case. There is a similar split in the theory of GWAs. One ingredient used in the construction of a GWA is an automorphism σ of its base ring. When σ has finite order, or when it acts with finite order on some maximal ideals of the base ring, there is a similar loss of "noncommutative rigidity" to that which occurs for quantized algebras at roots of unity. This work is focused on applications to *generic* quantized algebras, so it develops the side of GWA theory that typically has σ acting with infinite order.

Outline The theory of GWAs is laid out in section 2, and the applications to specific algebras are given in section 3. Sections 2.1 through 2.5 build up the needed background and notation, some of which is a collection of results that could be found in [4], [5], [6], and [14]. A description of homogeneous ideals of GWAs is given in section 2.2, and localization is explored in section 2.3. Section 2.4 addresses GK dimension by transporting the arguments of [29] for skew Laurent rings into the GWA setting. It is shown in Theorem 28 that the GK dimension of a GWA is one more than the GK dimension of its base ring, given a certain assumption on σ. Section 2.5 explores some of the finite dimensional representation theory of GWAs, focusing on the setting that will apply to generic quantized algebras.

Section 2.6 develops a powerful tool for working with the noncommutative prime spectrum of a GWA. Theorem 70 says that under some rather specific conditions, every prime ideal of a GWA arises as the annihilator of one of the simple modules from the classification of section 2.5. While the hypotheses of this theorem look restrictive at first glance, it has wide applicability due to the approach that it suggests: given a GWA, use quotients and localizations to partition its prime spectrum into pieces to which Theorem 70 can be applied. The theory of section 2.6 is demonstrated on a host of examples in section 3.

The most completely worked out example, to which the entirety of section 2 is applied, is the 2×2 reflection equation algebra $\mathcal{A}_q(\mathrm{M}_2)$. (The reason is that this algebra was the case study that launched the author's work on GWAs.) Normal elements are identified and then used to compute the automorphism group in Theorem 89. Theorem 90 provides a classification of the finite dimensional simple $\mathcal{A}_q(\mathrm{M}_2)$-modules that are not annihilated by the element u_{22}. We also find in Theorem 92 that finite dimensional $\mathcal{A}_q(\mathrm{M}_2)$-modules on which u_{22} acts invertibly are semisimple. The prime spectrum of $\mathcal{A}_q(\mathrm{M}_2)$ is fully worked out in section 3.1.4, and some consequences are explored. The prime spectrum appears in Theorem 105, and the primitive spectrum appears in Theorem 113, where it is shown that $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence. The semiclassical limit of $\mathcal{A}_q(\mathrm{M}_2)$ is studied in section 3.1.6, and the concept of a Poisson GWA is developed in section 2.7 to aid this study.

The initial determination of the prime spectrum of $\mathcal{A}_q(\mathrm{M}_2)$ was done without the general theory of section 2.6. Section 3.1.5 provides the simpler approach afforded by Theorem 70.

Notation All rings are rings with 1, and they are not necessarily commutative. The symbol k always denotes a field. Given a ring R and an automorphism σ of R, we use $R[x;\sigma]$ to denote the skew polynomial ring and $R[x^{\pm};\sigma]$ to denote the skew Laurent ring. Our convention for the twisting is such that $xr = \sigma(r)x$ for $r \in R$. If there is further twisting by a σ-derivation δ, then the notation becomes $R[x;\sigma,\delta]$. We will also use $R((x^{\pm};\sigma))$ to denote the skew Laurent series ring. Given a subset $\mathcal{G}$ of a ring R, we indicate by $\langle\mathcal{G}\rangle$ the two-sided ideal of R generated by $\mathcal{G}$. When there is some ambiguity as to the ring in which ideal generation takes place, we resolve it by using a subscript $\langle\mathcal{G}\rangle_R$ or by writing $\langle\mathcal{G}\rangle \triangleleft R$.

Basic Definitions A *regular* element of a ring is an element that is neither a left nor a right zero divisor. A *normal* element of a ring R is an element r for which $rR = Rr$. A *prime ideal* of a ring R is a proper ideal $P \triangleleft R$ with the property that $IJ \subseteq P \Rightarrow (I \subseteq P \text{ or } J \subseteq P)$ for all $I, J \triangleleft R$. For $I \triangleleft R$, define $V(I)$ to be the set of prime ideals of R that contain I. The *prime spectrum* of R, denoted by $\mathrm{spec}(R)$, is defined to be the set of prime ideals of R. The family $\{V(I) \mid I \triangleleft R\}$ forms the collection of closed sets of the *Zariski topology*; we consider $\mathrm{spec}(R)$ to be a topological space in this way. The subspace of $\mathrm{spec}(R)$ consisting of only the maximal ideals is called the *maximal spectrum*, and it is denoted by $\max\mathrm{spec}(R)$. The subspace of $\mathrm{spec}(R)$ consisting of only the primitive ideals is called the *primitive spectrum*, and it is denoted by $\mathrm{prim}(R)$. If R is a graded ring, then the subset of $\mathrm{spec}(R)$ consisting of only the homogeneous prime ideals is called the *graded spectrum*, and it is denoted by $\text{gr-spec}(R)$. Given an ideal I of a commutative ring R, we use $\hat{V}(I)$ to denote the set of *maximal* ideals containing I. (This is to avoid confusion with $V(I)$, the set of *prime* ideals containing I.) If $S \subseteq \max\mathrm{spec}(R)$ or $S \subseteq \mathrm{spec}(R)$, we denote by $I(S)$ the intersection $\bigcap S$ (the "ideal of S").

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2 Generalized Weyl Algebras

We shall define GWAs by presenting them as rings over a given base ring. A ring S *over* a ring R, also known as an R*-ring*, is simply a ring homomorphism $R \to S$. A morphism $S \to S'$ of rings over R is a ring homomorphism such that

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. Given any set $\mathcal{X}$, one can show that a *free* R-ring on $\mathcal{X}$ exists. This provides meaning to the notion of a presentation of a ring over R; it can be thought of as a ring over R satisfying a universal property described in terms of the relations.

Definition 1: Let R be a ring, σ an automorphism of R, and z an element of the center of R. The GWA based on this data is the ring over R generated by x and y subject to the relations

$$\begin{array}{lll} yx = z & xy = \sigma(z) & \\ xr = \sigma(r)x & yr = \sigma^{-1}(r)y & \forall\ r \in R. \end{array} \tag{1}$$

We denote this construction by

$$R[x, y; \sigma, z]$$

and we adapt some useful notation from [4] as follows. Define

$$v_n = \begin{cases} x^n & n \geq 0 \\ y^{(-n)} & n \leq 0 \end{cases}$$

for $n \in \mathbb{Z}$, and define

$$\sigma^{[j,k]}(z) = \prod_{l=j}^{k} \sigma^l(z)$$

for integers $j \leq k$. We take a product over an empty index set to be 1. Define the following special elements of $Z(R)$:

$$[\![n, m]\!] = \begin{cases} \sigma^{[n+m+1,n]}(z) & n > 0,\ m < 0,\ |n| \geq |m| \\ \sigma^{[1,n]}(z) & n > 0,\ m < 0,\ |n| \leq |m| \\ \sigma^{[n+1,n+m]}(z) & n < 0,\ m > 0,\ |n| \geq |m| \\ \sigma^{[n+1,0]}(z) & n < 0,\ m > 0,\ |n| \leq |m| \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

for $n, m \in \mathbb{Z}$. Now we have $v_n v_m = [\![n, m]\!] v_{n+m}$ for $n, m \in \mathbb{Z}$.

2.1 Basic Properties

This section lays down some basic ring-theoretic properties of GWAs, ones which we will need to reference in later sections. The following two propositions are easy observations.

Proposition 2: *There is an R-ring homomorphism $\phi : R[x, y; \sigma, z] \to R[x^{\pm}; \sigma]$ sending x to x and y to zx^{-1}. There is also an R-ring homomorphism $\phi' : R[x, y; \sigma, z] \to R[x^{\pm}; \sigma]$ sending x to xz and y to x^{-1}.*

Proof: Observe that the skew Laurent ring $R[x^{\pm}; \sigma]$ is an R-ring with x and zx^{-1} satisfying the defining relations for the GWA, and similarly for xz and x^{-1}. ■

Proposition 3: *$R[x, y; \sigma, z]$ has the alternative expression $R[y, x; \sigma^{-1}, \sigma(z)]$, and $R[x, y; \sigma, z]^{op}$ can be expressed as $R^{op}[x, y; \sigma^{-1}, \sigma(z)]$.*

Proof: Check, carefully, that the GWA relations hold where needed. ■

Proposition 3 roughly means that if we prove something (ring-theoretic) about x, then we get a y version of the result by swapping x and y, replacing σ with σ^{-1}, and replacing z with $\sigma(z)$. And if we prove something left-handed, then we get a right-handed version of the result by replacing σ with σ^{-1} and replacing z with $\sigma(z)$.

Proposition 4: *Consider a GWA $R[x, y; \sigma, z]$.*

1. *It is a free left (right) R-module on $\{v_i \mid i \in \mathbb{Z}\}$.*
2. *It has a $\mathbb{Z}$-grading with homogeneous components $Rv_n = v_nR$:*

$$R[x, y; \sigma, z] = \bigoplus_{n\in\mathbb{Z}} Rv_n$$

3. *It contains a copy of the ring R as the subring of degree zero elements. The subring generated by R and x is a skew polynomial ring $R[x; \sigma]$, and the subring generated by R and y is $R[y; \sigma^{-1}]$.*
4. *It is left (right) noetherian if R is left (right) noetherian.*

Proof: See [38, Lemma II.3.1.6] for a proof of assertion 1. Assertions 2 and 3 are then easily shown. For assertion 4 let S be the subring generated by R and x. Observe that $R[x, y; \sigma, z]$ is an over-ring of S generated by S and y such that

$$Sy + S = yS + S. \tag{3}$$

By the skew Hilbert basis theorem, $S = R[x; \sigma]$ is left (right) noetherian if R is. Using (3), one can write a version of the Hilbert basis theorem that applies to $R[x, y; \sigma, z]$ over S; see for example [32, Theorem 2.10]. That is, $R[x, y; \sigma, z]$ is left (right) noetherian if S is. ■

The following results are now routine.

Proposition 5: *Let $W = R[x, y; z, \sigma]$ be a GWA. The homomorphisms of Proposition 2 are injective if and only if $z \in R$ is regular, and they are isomorphisms if and only if $z \in R$ is a unit.*

Proof: Assume that z is regular. Let ϕ, ϕ' be as in 2. Consider any $w = \sum_{i\in\mathbb{Z}} a_iv_i \in W$ and assume that $\phi(w) = 0$. Then

$$0 = \phi(w) = \sum_{i\geq 0} a_ix^i + \sum_{i<0} a_iz^{-i}x^i,$$

so all a_i must vanish, since z is regular. Thus ϕ is injective. Assume for the converse that ϕ is injective. Then for any $a \in R$,

$$az = 0 \;\Leftrightarrow\; 0 = azx^{-1} = \phi(ay) \;\Leftrightarrow\; ay = 0 \;\Leftrightarrow\; a = 0,$$

so z is regular.

If z is a unit, then ϕ is injective by the above, and it is surjective because its image contains x and x^{-1}. Assume for the converse that ϕ is an isomorphism. Then $x^{-1} = \phi(w)$ for some $w \in W$. Clearly $w = ay$ for some $a \in R$. Now $x^{-1} = \phi(ay) = (az)x^{-1}$, so $az = 1$.

The proof regarding ϕ' is similar. (Actually, ϕ and ϕ' are related via Proposition 3 and the isomorphism $R[x^{\pm}; \sigma] \cong R[x^{\pm}; \sigma^{-1}]$ sending $x \mapsto x^{-1}$. So the result for ϕ' follows from symmetry.) ■

Corollary 6: *A GWA $W = R[x, y; \sigma, z]$ is a domain if and only if R is a domain and $z \neq 0$.*

Proof: Proposition 5 shows that if R is a domain and $z \neq 0$ then W embeds into the domain $R[x^{\pm}; \sigma]$. The converse follows from Proposition 4 and the fact that $yx = z$. ■

Proposition 7: *Let $W = R[x, y; \sigma, z]$ be a GWA. Then $x, y \in W$ are regular if and only if $z \in R$ is regular.*

Proof: If x and y are regular in W, then so is $yx = z$. Suppose for the converse that z is regular in R. By Proposition 5, W can be considered to be a subring of $R[x^{\pm}; \sigma]$ with $y = zx^{-1}$. Since $x \in R[x^{\pm}; \sigma]$ is regular, we conclude that x and y are regular. ■

The center of a GWA is often easily described when its coefficient ring is a domain:

Proposition 8: *Let R be a domain, and let σ be an automorphism of R such that $\sigma|_{Z(R)} : Z(R) \to Z(R)$ has infinite order. Then $Z(R[x, y; \sigma, z])$ is $Z(R)^{\sigma}$, the subring of $Z(R)$ fixed by σ.*

Proof: If $a \in Z(R)^{\sigma}$, then a commutes with R, x, and y and is therefore central. Suppose for the converse that $a = \sum_{m \in \mathbb{Z}} a_m v_m$ is central. Then $xa = ax$ and $ya = ay$ require that $\sigma(a_m) = a_m$ for all $m \in \mathbb{Z}$. Given any nonzero $m \in \mathbb{Z}$, our hypothesis ensures that there is some $r \in Z(R)$ such that $\sigma^m(r) \neq r$. Now $ra = ar$ requires $ra_m = a_m \sigma^m(r)$, so $a_m = 0$. Thus $a = a_0 \in R^{\sigma}$. Finally, a commutes with R, so $a \in Z(R)^{\sigma}$. ■

There are similar and easily verified facts about skew Laurent polynomials and skew Laurent series:

Proposition 9: *Let R be a domain, and let σ be an automorphism of R such that $\sigma|_{Z(R)} : Z(R) \to Z(R)$ has infinite order. Then $Z(R[x^{\pm}; \sigma]) = Z(R)^{\sigma}$, the subring of $Z(R)$ fixed by σ. Similarly, $Z(R((x^{\pm}; \sigma))) = Z(R)^{\sigma}$.*

Under some stronger conditions, one can also characterize the normal elements of a GWA:

Proposition 10: *Let R be a domain, and let σ be an automorphism of R such that there is an $r \in Z(R)$ which is not fixed by any nonzero power of σ. Then the normal elements of $W = R[x, y; \sigma, z]$ are homogeneous.*

Proof: Suppose that $a = \sum a_m v_m \in W$ is a nonzero normal element. Then $ra = ab$ for some $b \in W$. Looking at the highest degree and lowest degree terms of ra, and considering that R is a domain, b must have degree 0 in order for ab to have the same highest and lowest degree terms as ra. Thus $b \in R$. Now $ra = ab$ becomes

$$ra_m = a_m \sigma^m(b)$$

for all $m \in \mathbb{Z}$. Since r is central, we may cancel the a_m whenever it is nonzero. If a_m is nonzero for multiple $m \in \mathbb{Z}$, then $r = \sigma^m(b) = \sigma^{m+n}(b)$ for some $m, n \in \mathbb{Z}$ with $n \neq 0$. But $r = \sigma^n(r)$ would contradict our assumption on r, so a must be homogeneous. ■

Proposition 11: *Let R be a commutative domain, σ an automorphism, and $z \in R$ such that $\sigma^m(z)$ is never a unit multiple of z for nonzero $m \in \mathbb{Z}$. Then the normal elements of $W = R[x, y; \sigma, z]$ are the $r \in R$ such that $\sigma(r)$ is a unit multiple of r.*

Proof: Suppose that $r \in R$ and $\sigma(r) = ur$, where $u \in R^{\times}$. Then $rR = Rr$ because R is commutative, $xr = r(ux)$, $rx = (u^{-1}x)r$, $yr = r(yu^{-1})$, and $ry = (yu)r$. Thus r is normal in W. Now assume for the converse that $a \in W$ is normal and nonzero. By Proposition 10, using the fact that z is not fixed by any nonzero powers of σ, a is homogeneous. Write it as $a = a_m v_m$.

Suppose that $m \geq 0$, so that $a = a_m x^m$. For some $b \in W$, $ax = ba$. Clearly b must have the form $b_1 x$ for some $b_1 \in R$, so we have $a_m = b_1\sigma(a_m)$. Thus $a_m R \subseteq \sigma(a_m)R$. For some $c \in W$, $xa = ac$. Then c must have the form $c = c_1 x$ for some $c_1 \in R$, so we have $\sigma(a_m) = a_m\sigma^m(c_1)$. Thus $\sigma(a_m)R \subseteq a_m R$. We conclude that $\sigma(a_m)R = a_m R$.

If $m \leq 0$, then we may use the $x \leftrightarrow y$ symmetry of Proposition 3 to apply the above argument and conclude that $\sigma^{-1}(a_m)R = a_m R$. So in either case, $\sigma(a_m) = ua_m$ for some $u \in R^\times$.

Suppose that $m > 0$, so that $a = a_m x^m$. For some $d \in W$, $ay = da$. Clearly d must have the form $d_{-1}y$ for some $d_{-1} \in R$, so we have

$$a_m\sigma^m(z) = d_{-1}\sigma^{-1}(a_m)z = d_{-1}\sigma^{-1}(u^{-1})a_m z.$$

Thus, cancelling the a_m, $\sigma^m(z)R \subseteq zR$. For some $e \in W$, $ae = ya$. Then e must have the form $e = e_{-1}y$ for some $e_{-1} \in R$, so we have

$$a_m\sigma^m(e_{-1})\sigma^m(z) = \sigma^{-1}(a_m)z = \sigma^{-1}(u^{-1})a_m z.$$

Thus, cancelling the a_m, $zR \subseteq \sigma^m(z)R$. We conclude that $zR = \sigma^m(z)R$, contradicting the hypothesis on z. Therefore one cannot have $m > 0$.

If $m < 0$, then we may use $x \leftrightarrow y$ symmetry to apply the above argument and conclude that $\sigma(z)R = (\sigma^{-1})^m(\sigma(z))R$. But this is equivalent to the contradiction $zR = \sigma^{-m}(z)R$, so one cannot have $m < 0$ either. Therefore $m = 0$, and $a = a_0 \in R$ with $\sigma(a) = ua$. ■

2.2 Ideals

We will establish in this section a notation for discussing the homogeneous ideals of a GWA. We will also explore a portion of the prime spectrum of a GWA. First, note that quotients by ideals in the coefficient ring work as they ought to:

Proposition 12: *Let $W = R[x, y; \sigma, z]$ be a GWA, with $J \triangleleft R$ an ideal such that $\sigma(J) = J$. Let $I \triangleleft W$ be generated by J. Then there is a canonical isomorphism*

$$W/I \cong (R/J)[x, y; \hat{\sigma}, z + J], \tag{4}$$

where $\hat{\sigma}$ is the automorphism of R/J induced by σ.

Proof: Extend

$$R \twoheadrightarrow R/J \hookrightarrow (R/J)[x, y; \hat{\sigma}, z + J]$$

to W by sending x to x and y to y and checking that the needed GWA relations hold. Since the kernel of the resulting map contains J, it contains the ideal I generated by J. This defines one direction of (4). For the other, observe that the kernel of

$$R \hookrightarrow W \twoheadrightarrow W/I$$

contains J, and pass to the induced map $R/J \to W/I$. Extend this to $(R/J)[x, y; \hat{\sigma}, z + J]$ by sending x to x and y to y and checking that the needed GWA relations hold. The two homomorphisms just defined are inverse isomorphisms. ■

We will generally abuse notation and reuse the labels "σ" and "z" instead of using $\hat{\sigma}$ or $z + J$.

Definition 13: Whenever I is a subset of a GWA $R[x, y; \sigma, z]$ and $m \in \mathbb{Z}$, I_m shall denote the subset

$$I_m := \{r \in R \mid rv_m \in I\}$$

of R and I_m^{op} shall denote

$$I_m^{\mathrm{op}} := \{r \in R \mid v_m r \in I\}.$$

Remark 14: I_m^{op} is a notational device for working with the symmetry $R[x,y;\sigma,z]^{\mathrm{op}} = R^{\mathrm{op}}[x,y;\sigma^{-1},\sigma(z)]$. It transfers the definition of I_m to the GWA structure on the opposite ring. Note that the relation is that $I_m^{\mathrm{op}} = \sigma^{-m}(I_m)$ for all $m \in \mathbb{Z}$.

Propositions 15 to 18 were essentially observed in [6].

Proposition 15: *Let I be a right $R[x;\sigma]$-submodule of $R[x,y;\sigma,z]$. The I_n are right ideals of R, and they satisfy*

$$I_{-(n+1)}\sigma^{-n}(z) \subseteq I_{-n} \qquad I_n \subseteq I_{n+1} \tag{5}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Thus, a homogeneous right $R[x;\sigma]$-submodule I of $R[x,y;\sigma,z]$ has the form $\bigoplus_{n\in\mathbb{Z}} I_n v_n$ for a family $(I_n)_{n\in\mathbb{Z}}$ of right ideals of R satisfying (5). Further, any such family $(I_n)_{n\in\mathbb{Z}}$ defines a right $R[x;\sigma]$-submodule of $R[x,y;\sigma,z]$ in this way.

Proof: Let I be a right $R[x;\sigma]$-submodule of $R[x,y;\sigma,z]$, and let $n \in \mathbb{Z}_{\geq 0}$. The I_n are right ideals of R because $rsv_n = rv_n\sigma^{-n}(s) \in I$ whenever $rv_n \in I$ and $s \in R$. If $a \in I_n$, then

$$I \ni (ax^n)x = ax^{n+1},$$

so $a \in I_{n+1}$. And if $a \in I_{-(n+1)}$, then $a\sigma^{-n}(z)y^n = ay^{n+1}x \in I$, so $a\sigma^{-n}(z) \in I_{-n}$. This establishes (5). For the final assertion, assume that $(I_n)_{n\in\mathbb{Z}}$ is a family of right ideals of R satisfying (5), and let $I = \bigoplus_{n\in\mathbb{Z}} I_n v_n$. The ring $R[x;\sigma]$ is generated by $R \cup \{x\}$, so one only needs to verify that I is stable under right multiplication by R and x. The former follows from the fact that the I_n are right ideals of R, and the latter is ensured by (5). ■

Proposition 16: *Let I be a right ideal of $R[x,y;\sigma,z]$. The I_n are right ideals of R, and they satisfy*

$$\begin{array}{ll} I_{-(n+1)} \supseteq I_{-n} & I_n \subseteq I_{n+1} \\ I_{-(n+1)}\sigma^{-n}(z) \subseteq I_{-n} & I_n \supseteq I_{n+1}\sigma^{n+1}(z) \end{array} \tag{6}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Thus, a homogeneous right ideal I of $R[x,y;\sigma,z]$ has the form $\bigoplus_{n\in\mathbb{Z}} I_n v_n$ for a family $(I_n)_{n\in\mathbb{Z}}$ of right ideals of R satisfying (6). Further, any such family $(I_n)_{n\in\mathbb{Z}}$ defines a right ideal of $R[x,y;\sigma,z]$ in this way.

Proof: To be a right ideal of $R[x,y;\sigma,z]$ is to be stable under the right multiplication by both of the subrings $R[x;\sigma]$ and $R[y;\sigma^{-1}]$. We shall extend the assertions of Proposition 15 using the symmetries indicated in Proposition 3. We take $R[y,x;\sigma^{-1},\sigma(z)]$ to be *equal* to the ring $R[x,y;\sigma,z]$, with our focus merely shifted to a different GWA structure. Applying Proposition 15 to $R[y,x;\sigma^{-1},\sigma(z)]$ is a matter of swapping x and y, replacing σ by σ^{-1}, and replacing z by $\sigma(z)$. The conditions in (5) become:

$$I_{n+1}\sigma^{n+1}(z) \subseteq I_n \qquad I_{-n} \subseteq I_{-n+1}. \tag{7}$$

Thus this Proposition is a consequence of Proposition 15 applied to both $R[x,y;\sigma,z]$ and $R[y,x;\sigma^{-1},\sigma(z)]$. ■

Proposition 17: *Let I be a left ideal of $R[x,y;\sigma,z]$. The I_n are left ideals of R, and they satisfy*

$$\begin{array}{ll} \sigma^{n+1}(I_{-(n+1)}) \supseteq \sigma^n(I_{-n}) & \sigma^{-n}(I_n) \subseteq \sigma^{-(n+1)}(I_{n+1}) \\ \sigma^{n+1}(I_{-(n+1)})\sigma^{n+1}(z) \subseteq \sigma^n(I_{-n}) & \sigma^{-n}(I_n) \supseteq \sigma^{-(n+1)}(I_{n+1})\sigma^{-n}(z) \end{array} \tag{8}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Thus, a homogeneous left ideal I of $R[x,y;\sigma,z]$ has the form $\bigoplus_{n\in\mathbb{Z}} I_n v_n$ for a family $(I_n)_{n\in\mathbb{Z}}$ of left ideals of R satisfying (8). Further, any such family $(I_n)_{n\in\mathbb{Z}}$ defines a left ideal of $R[x,y;\sigma,z]$ in this way.

Proof: We again take advantage of the symmetries indicated in Proposition 3. Applying Proposition 16 to $R[x, y; \sigma, z]^{\mathrm{op}} = R^{\mathrm{op}}[x, y; \sigma^{-1}, \sigma(z)]$ is a matter of replacing σ by σ^{-1}, and replacing z by $\sigma(z)$. The conditions in (6) become:

$$\begin{array}{ll} I^{\mathrm{op}}_{-(n+1)} \supseteq I^{\mathrm{op}}_{-n} & \qquad I^{\mathrm{op}}_n \subseteq I^{\mathrm{op}}_{n+1} \\ I^{\mathrm{op}}_{-(n+1)}\sigma^{n+1}(z) \subseteq I^{\mathrm{op}}_{-n} & \qquad I^{\mathrm{op}}_n \supseteq I^{\mathrm{op}}_{n+1}\sigma^{-n}(z), \end{array} \tag{9}$$

Making the adjustment in Remark 14 to (9) yields (8). Thus this proposition is a consequence of Proposition 16 applied to $R^{\mathrm{op}}[x, y; \sigma^{-1}, \sigma(z)]$. ■

Proposition 18: *Let I be an ideal of $R[x, y; \sigma, z]$. The I_n are ideals of R, and they satisfy (6) and (8) for all $n \in \mathbb{Z}_{\geq 0}$. Thus, a homogeneous ideal I of $R[x, y; \sigma, z]$ has the form $\bigoplus_{n\in\mathbb{Z}} I_n v_n$ for a family $(I_n)_{n\in\mathbb{Z}}$ of ideals of R satisfying (6) and (8). Further, any such family $(I_n)_{n\in\mathbb{Z}}$ defines an ideal of $R[x, y; \sigma, z]$ in this way.*

Proof: Use Propositions 16 and 17. ■

We may depict (6) and (8) by the following diagrams:

$$\begin{tikzcd}[column sep=small] \cdots & \supseteq & I_{-2} \arrow[rr, bend right, "\sigma^{-1}(z)"'] & \supseteq & I_{-1} \arrow[rr, bend right, "z"'] & \supseteq & I_0 & \subseteq & I_1 \arrow[ll, bend left, "\sigma(z)"] & \subseteq & I_2 \arrow[ll, bend left, "\sigma^2(z)"] & \subseteq & \cdots \\ \cdots & \supseteq & \sigma^2(I_{-2}) \arrow[rr, bend right, "\sigma^2(z)"'] & \supseteq & \sigma(I_{-1}) \arrow[rr, bend right, "\sigma(z)"'] & \supseteq & I_0 & \subseteq & \sigma^{-1}(I_1) \arrow[ll, bend left, "z"] & \subseteq & \sigma^{-2}(I_2) \arrow[ll, bend left, "\sigma^{-1}(z)"] & \subseteq & \cdots . \end{tikzcd}$$

We may also depict an alternative way of stating (8),

$$\begin{array}{ll} I_{-(n+1)} \supseteq \sigma^{-1}(I_{-n}) & \qquad \sigma(I_n) \subseteq I_{n+1} \\ \sigma(I_{-(n+1)})\sigma(z) \subseteq I_{-n} & \qquad I_n \supseteq \sigma^{-1}(I_{n+1})z, \end{array}$$

by the following diagram:

$$\begin{tikzcd} \cdots & I_{-2} \arrow[r, bend right, "{\sigma, \sigma(z)}"'] & I_{-1} \arrow[l, bend right, "\sigma^{-1}"'] \arrow[r, bend right, "{\sigma, \sigma(z)}"'] & I_0 \arrow[l, bend right, "\sigma^{-1}"'] \arrow[r, bend left, "\sigma"] & I_1 \arrow[l, bend left, "{\sigma^{-1}, z}"] \arrow[r, bend left, "\sigma"] & I_2 \arrow[l, bend left, "{\sigma^{-1}, z}"] & \cdots . \end{tikzcd}$$

Lemma 20 below will be useful for working out the prime spectrum for certain GWAs. We first establish the following proposition, which identifies one situation in which the upcoming condition (13) of Lemma 20 holds for a given family of ideals.

Proposition 19: *Let $A \subseteq B$ be rings such that B is a free left A-module with a basis $(b_j)_{j\in\mathcal{J}}$ for which $Ab_j = b_jA$ for all $j \in \mathcal{J}$. Let $(I_\alpha)_{\alpha\in\mathfrak{A}}$ be a family of ideals of A satisfying $b_jI_\alpha \subseteq I_\alpha b_j$ for all j and α. Then*

$$\bigcap_{\alpha\in\mathfrak{A}} BI_\alpha B = B\left(\bigcap_{\alpha\in\mathfrak{A}} I_\alpha\right)B. \tag{10}$$

Proof: We begin by showing that $b_j\left(\bigcap_\alpha I_\alpha\right) \subseteq \left(\bigcap_\alpha I_\alpha\right)b_j$ for all j. Consider any $j \in \mathcal{J}$ and any $r \in \bigcap_\alpha I_\alpha$. There is, for each $\alpha \in \mathfrak{A}$, an $r'_\alpha \in I_\alpha$ such that $b_jr = r'_\alpha b_j$. Since b_j came from a basis for ${}_AB$, all the r'_α are equal, and so we've shown that $b_j\left(\bigcap_\alpha I_\alpha\right) \subseteq \left(\bigcap_\alpha I_\alpha\right)b_j$ for all j.

Let I be any ideal of A satisfying $b_jI \subseteq Ib_j$ for all j. Observe that $\bigoplus_{j\in\mathcal{J}} Ib_j$ is then an ideal of B, and hence it is the extension of I to an ideal of B. Applying this principle to $I = I_\alpha$ for $\alpha \in \mathfrak{A}$, and also applying it to $I = \bigcap_\alpha I_\alpha$, (10) follows from the fact that

$$\bigcap_{\alpha\in\mathfrak{A}} \left(\bigoplus_{j\in\mathcal{J}} I_\alpha b_j \right) = \bigoplus_{j\in\mathcal{J}} \left(\bigcap_{\alpha\in\mathfrak{A}} I_\alpha \right) b_j. \qquad \blacksquare$$

Lemma 20: *Let $W = R[x, y; \sigma, z]$ be a GWA such that $R^\sigma \subseteq R$ has the following property:*

$$\forall I \lhd R^\sigma,\ RIR \cap R^\sigma = I. \tag{11}$$

Then there are mutually inverse inclusion-preserving bijections

$$\begin{array}{ccc} \{I \mid I \lhd R^\sigma\} & \leftrightarrow & \{WIW \mid I \lhd R^\sigma\} \\ I & \mapsto & WIW \\ \mathcal{I} \cap R^\sigma & \leftarrow\!\shortmid & \mathcal{I}. \end{array} \tag{12}$$

Now let $S = \{W\mathfrak{p}W \mid \mathfrak{p} \in \operatorname{spec}(R^\sigma)\}$ and assume that $S \subseteq \operatorname{spec}(W)$. Assume also that extension of ideals to R preserves intersections in the following sense: for any family $(I_\alpha)_{\alpha\in\mathfrak{A}}$ of ideals of R^σ,

$$\bigcap_{\alpha\in\mathfrak{A}} RI_\alpha R = R(\bigcap_{\alpha\in\mathfrak{A}} I_\alpha)R. \tag{13}$$

Then (12) restricts to a homeomorphism

$$\operatorname{spec}(R^\sigma) \approx S.$$

Proof: Given any $I \lhd R^\sigma$,

$$WIW = \bigoplus_{m\in\mathbb{Z}} RIR\, v_m, \tag{14}$$

because the right hand side satisfies the conditions of Proposition 18 needed to make it an ideal of W. Given $I, J \lhd R^\sigma$ with $WIW \subseteq WJW$, we have $RIR \subseteq RJR$ from looking at the degree zero component. From (11) we can then deduce that $I \subseteq J$. The converse of this is clear: $I \subseteq J \Rightarrow WIW \subseteq WJW$. Putting this information together, we have the inclusion-preserving correspondence (12).

Now assume that $S \subseteq \operatorname{spec}(W)$ and that (13) holds. Let $\phi : \operatorname{spec}(R^\sigma) \to S$ be the restriction of (12). We show that the bijection ϕ is a homeomorphism.

ϕ is a closed map: Given any $I \lhd R^\sigma$, one has that $\mathfrak{p} \supseteq I$ if and only if $W\mathfrak{p}W \supseteq WIW$, for all $\mathfrak{p} \in \operatorname{spec}(R^\sigma)$. That is, the collection of $\mathfrak{p} \in \operatorname{spec}(R^\sigma)$ that contain I is mapped by ϕ onto the collection of $P \in S$ that contain WIW.

ϕ is continuous: Let $K \lhd W$. Define $\mathcal{J} := \{J \lhd R^\sigma \mid K \subseteq WJW\}$ and $I := \bigcap \mathcal{J}$ (with an intersection of the empty set being R^σ). For $\mathfrak{p} \in \operatorname{spec}(R^\sigma)$, if $W\mathfrak{p}W \supseteq K$, then $\mathfrak{p} \in \mathcal{J}$, so $\mathfrak{p} \supseteq I$. And if $I \subseteq \mathfrak{p}$, then

$$K \subseteq \bigcap_{J\in\mathcal{J}} WJW = W\left(\bigcap_{J\in\mathcal{J}} RJR\right)W = WIW \subseteq W\mathfrak{p}W,$$

where the first equality is an application of Proposition 19 to $R \subseteq W$, and the second equality is due to the assumption (13). We have therefore shown that the collection of $P \in S$ that contain K pulls back via ϕ to the collection of $\mathfrak{p} \in \operatorname{spec}(R^\sigma)$ that contain I. $\blacksquare$

2.3 Localizations

Proposition 21: *Let $W = R[x, y; \sigma, z]$ be a GWA with z regular. Then $\mathcal{S} := \{1, x, x^2, \dots\}$ is an Ore set of regular elements, and the corresponding ring of fractions is given by the homomorphism $\phi : W \to R[x^{\pm}; \sigma]$ of Proposition 2.*

Proof: That the elements of $\mathcal{S}$ are regular comes from Proposition 7. If we can show that ϕ is the localization homomorphism for a right ring of fractions of W with respect to $\mathcal{S}$, then by [21, Theorem 6.2] we will have that $\mathcal{S}$ is a right Ore set. Then it will also be a left Ore set due to Proposition 3, of course with the same ring of fractions, by [21, Proposition 6.5].

So we have only two things to verify: that $\phi(\mathcal{S})$ is a collection of units and that elements of $R[x^{\pm}; \sigma]$ have the form $\phi(w)\phi(s)^{-1}$ with $w \in W$ and $s \in S$. The former statement is obvious. For the latter, consider an arbitrary $p \in R[x^{\pm}; \sigma]$. There is some $n \in \mathbb{Z}_{\geq 0}$ such that $px^n \in R[x; \sigma]$. Observe that, by Proposition 5, ϕ maps the R-subring $R[x; \sigma]$ of W generated by x isomorphically to the R-subring $R[x; \sigma]$ of $R[x^{\pm}; \sigma]$. So

$$p = \phi(\phi^{-1}(px^n))\phi(x^n)^{-1},$$

proving that ϕ gives a right ring of fractions. That ϕ also works as a left ring of fractions then follows from Proposition 3. ■

Proposition 22: *Let $W = R[x, y; \sigma, z]$ be a GWA. Let $\mathcal{S} \subseteq R$ be a right denominator set, and assume that $\sigma(\mathcal{S}) = \mathcal{S}$. Then $\mathcal{S}$ is a right denominator set of W, and the associated localization map has the following description: Let $\phi_0 : R \to R\mathcal{S}^{-1}$ be the localization map for the right ring of fractions of R. Let $\hat{\sigma}$ be the automorphism of $R\mathcal{S}^{-1}$ induced by σ, and let $\hat{z} = \phi_0(z)$. Let $\phi : W \to R\mathcal{S}^{-1}[x, y; \hat{\sigma}, \hat{z}]$ be the extension of $R \xrightarrow{\phi_0} R\mathcal{S}^{-1} \hookrightarrow R\mathcal{S}^{-1}[x, y; \hat{\sigma}, \hat{z}]$ to W that sends x to x and y to y. This is the desired localization map. In short,*

$$W\mathcal{S}^{-1} \;=\; (R\mathcal{S}^{-1})[x, y; \hat{\sigma}, \hat{z}].$$

An analogous statement holds for left denominator sets.

Proof: Note that $\hat{\sigma}$ exists due to our hypothesis $\sigma(\mathcal{S}) = \mathcal{S}$. And the extension ϕ of ϕ_0 exists because GWA relations hold where needed. If we can show that ϕ really does define a right ring of fractions of $R[x, y; \sigma, z]$ with respect to $\mathcal{S}$, then it will follow that $\mathcal{S}$ is a right denominator set in $R[x, y; \sigma, z]$ (by [21, Theorem 10.3] for example). Thus, three things need to be verified: that $\phi(\mathcal{S})$ is a collection of units, that elements of $R\mathcal{S}^{-1}[x, y; \hat{\sigma}, \hat{z}]$ have the form $\phi(w)\phi(s)^{-1}$ with $w \in W$ and $s \in S$, and that the kernel of ϕ is $\{w \in W \mid ws = 0 \text{ for some } s \in \mathcal{S}\}$. That $\phi(\mathcal{S})$ is a collection of units is obvious.

Let $\sum_{i \in \mathbb{Z}} a_i v_i$ be an arbitrary element of $R\mathcal{S}^{-1}[x, y; \hat{\sigma}, \hat{z}]$. Get a "common right denominator" $s \in \mathcal{S}$ and elements r_i of R so that $a_i = \phi_0(r_i)\phi_0(s)^{-1}$ for all $i \in \mathbb{Z}$ (see [21, Lemma 10.2a], noting that all but finitely many of the a_i vanish). Then

$$\begin{aligned}\sum a_i v_i = \sum \phi_0(r_i)\phi_0(s)^{-1} v_i = \sum \phi_0(r_i) v_i \hat{\sigma}^{-i}(\phi_0(s)^{-1}) = \sum \phi(r_i v_i)\phi_0(\sigma^{-i}(s))^{-1} \\ = \sum \phi(r_i v_i)\phi(\sigma^{-i}(s))^{-1}.\end{aligned}$$

After a further choice of common denominator, we see that $\sum_{i \in \mathbb{Z}} a_i v_i$ has the needed form. It remains to examine the kernel of ϕ. Let $w = \sum r_i v_i$ be an arbitrary element of W. If $ws = 0$ with $s \in \mathcal{S}$, then $\phi(w)$ must vanish because $\phi(s)$ is a unit. Assume for the converse that $0 = \phi(w) = \sum \phi_0(r_i) v_i$. Then $r_i \in \ker(\phi_0)$ for all i, so there are $s_i \in \mathcal{S}$ such that $r_i s_i = 0$ for all i. By [21, Lemma 4.21], there are $b_i \in R$ such that the products $\sigma^{-i}(s_i) b_i$ are all equal to a single $s \in \mathcal{S}$. Then

$$ws = \sum r_i v_i \sigma^{-i}(s_i) b_i = \sum r_i s_i v_i b_i = 0.$$

Thus $\ker(\phi) = \{w \in W \mid ws = 0 \quad \text{for some } s \in \mathcal{S}\}$, and this completes the proof of the right-handed version of the theorem. The left-handed version then follows from Proposition 3. ■

We will generally abuse notation and reuse the labels "σ" and "z."

Corollary 23: *The localization of $W = R[x, y; \sigma, z]$ at the multiplicative set $\mathcal{S}$ generated by $\{\sigma^i(z) \mid i \in \mathbb{Z}\}$ is a skew Laurent ring $(R\mathcal{S}^{-1})[x^{\pm}; \sigma]$, where the localization map extends the one $R \to R\mathcal{S}^{-1}$ by sending x to x and y to zx^{-1}.*

Proof: Use Proposition 22 to describe the localization. Then observe that it is isomorphic to a skew Laurent ring by Proposition 5, since z has become a unit. ■

We will often use localization to adjust the base ring of a GWA so that it becomes easier to determine prime ideals. Prime ideals can then be pulled back to the original GWA, but actually describing them in terms of generators can be tricky. The following lemma can aid this situation for homogeneous primes with a commutative base ring.

Lemma 24: *Let $W = R[x, y; \sigma, z]$ be a GWA over a commutative noetherian ring R. Let $\mathcal{S} \subseteq R$ be a denominator set such that $\sigma(\mathcal{S}) = \mathcal{S}$. Using Proposition 22, identify the localization $W\mathcal{S}^{-1}$ with the GWA $R\mathcal{S}^{-1}[x, y; \sigma, z]$. Let $P = \bigoplus_{m\in\mathbb{Z}} P_m v_m$ be a homogeneous prime ideal of $W\mathcal{S}^{-1}$. Then the contraction P^c of P to W can be described as*

$$P^c = \bigoplus_{m\in\mathbb{Z}} P_m^c v_m,$$

where each P_m^c is the contraction of the ideal $P_m \lhd R\mathcal{S}^{-1}$ to R.

Proof: Let $\phi : W \to R\mathcal{S}^{-1}[x, y; \sigma(z)]$ denote the localization map. Define $G = \bigoplus_{m\in\mathbb{Z}} P_m^c v_m$. By Proposition 18, the family $(P_m)_{m\in\mathbb{Z}}$ satisfies the conditions (6) and (8). Note that the σ that appears in these expressions is the automorphism of $R\mathcal{S}^{-1}$ *induced* by our present $\sigma \in \mathrm{Aut}(R)$, and the z that appears is the *image* of our present z in $R\mathcal{S}^{-1}$. With this in mind, it follows easily that the conditions (6) and (8) hold for the family $(P_m^c)_{m\in\mathbb{Z}}$. Thus G is a two-sided ideal of W, by Proposition 18.

We have $\phi(G) = \bigoplus_{m\in\mathbb{Z}} \phi(P_m^c) v_m$, so since $\phi(P_m^c) \subseteq P_m$ for all $m \in \mathbb{Z}$, we have $\langle \phi(G) \rangle_W \subseteq P$. We also have $\langle \phi(P_m^c) \rangle_{R\mathcal{S}^{-1}} v_m \subseteq \langle \phi(G) \rangle_W$. For each $m \in \mathbb{Z}$, the ideal P_m is the extension of its contraction: $P_m = \langle \phi(P_m^c) \rangle_{R\mathcal{S}^{-1}}$. So $P_m v_m \subseteq \langle \phi(G) \rangle_W$ for all $m \in \mathbb{Z}$. It follows that $\langle \phi(G) \rangle_W = P$.

The goal is now to apply Lemma 143 (in Appendix A) to conclude that G is P^c, but we must verify its hypotheses 2 and 3. We will use a left-handed version of Lemma 143. For hypothesis 2, we will show that ${}_W(W/G)$ is $\mathcal{S}$-torsionfree. Given an $s \in \mathcal{S}$ and $w = \sum_{m\in\mathbb{Z}} r_m v_m \in W$ such that $sw \in G$, we get $sr_m \in P_m^c$ for all $m \in \mathbb{Z}$. Hence the problem is reduced to showing that ${}_R(R/P_m^c)$ is $\mathcal{S}$-torsionfree for each $m \in \mathbb{Z}$. But this follows immediately from the fact that each P_m^c is a contraction of an ideal of $R\mathcal{S}^{-1}$.

For hypothesis 3 of Lemma 143, consider any $g \in G$ and $s \in \mathcal{S}$. We must show that $\mathcal{S}g \cap Gs$ is nonempty. Write g as $\sum_{m\in J} r_m v_m$, where $J \subseteq \mathbb{Z}$ is a finite index set and $r_m \in P_m^c$ for $m \in J$. Let $s' = \prod_{m\in J} \sigma^m(s)$ and let $r'_m = r_m \prod_{n\in J\setminus\{m\}} \sigma^n(s)$ for $m \in J$. Then it is clear that $s' \in \mathcal{S}$ and $\sum_{m\in J} r'_m v_m \in G$. Further, we have

$$s'g = \sum_{m\in J} s' r_m v_m = \sum_{m\in J} \sigma^m(s) r'_m v_m = \left(\sum_{m\in J} r'_m v_m \right) s \in \mathcal{S}g \cap Gs.$$

Thus Lemma 143 applies and $G = P^c$. ■

2.4 Gelfand-Kirillov Dimension

Throughout this section, R denotes an algebra over a field k, z a central element, $\sigma : R \to R$ an algebra automorphism, and W the GWA $R[x, y; \sigma, z]$.

Definition 25: If A is a finitely generated k-algebra then we define its *Gelfand-Kirillov dimension* $\mathrm{GK}(A)$ to be

$$\limsup \log_n \dim(V^n),$$

where V is any choice of finite dimensional generating subspace for A such that $1 \in V$. If A is not finitely generated, then we take $\mathrm{GK}(A)$ to be the supremum of $\mathrm{GK}(A')$ for all affine subalgebras A' of A. For more details and to see that Gelfand-Kirillov dimension is well defined, see [32, Chapter 8] or [27].

Proposition 26: $\mathrm{GK}(W) \geq \mathrm{GK}(R) + 1$

Proof: Since W contains a copy of the skew polynomial ring $R[x; \sigma]$, the problem reduces to showing that $\mathrm{GK}(R[x; \sigma]) \geq \mathrm{GK}(R) + 1$. The proof is standard (c.f. [27, Lemma 3.4]). We provide it for completeness.

Let A be any affine subalgebra of R. Let V be a finite dimensional generating subspace for A with $1 \in V$. Let $X = V + kx$. Then for $n \geq 1$,

$$X^{2n} = (V + kx)^{2n} \supseteq V^n + V^n x + \cdots + V^n x^n,$$

so $\dim(X^{2n}) \geq (n+1)\dim(V^n)$. Thus,

$$\begin{aligned} \mathrm{GK}(W) &\geq \limsup \log_n \dim(X^{2n}) \\ &\geq \limsup \log_n((n+1)\dim(V^n)) \\ &= \lim_{n\to\infty} \log_n(n+1) + \limsup \log_n(\dim(V^n)) \\ &= 1 + \mathrm{GK}(A). \end{aligned}$$

Since A was an arbitrary affine subalgebra of R, this gives $\mathrm{GK}(W) \geq \mathrm{GK}(R) + 1$. ■

Under what conditions can Proposition 26 be upgraded to an equality? We look to the skew Laurent case, i.e. the case in which z is a unit, for some guidance.

Definition 27: An algebra automorphism $\sigma : R \to R$ is *locally algebraic* if and only if for each $r \in R$, $\{\sigma^n(r) \mid n \geq 0\}$ spans a finite dimensional subspace of R. Equivalently, σ is locally algebraic if and only if every finite dimensional subspace of R is contained in some σ-stable finite dimensional subspace of R.

It was shown in [29, Prop. 1] that if σ is locally algebraic, then $\mathrm{GK}(R[x^{\pm}; \sigma]) = \mathrm{GK}(R) + 1$. The locally algebraic assumption was also shown to be partly necessary in [41], for example when R is a commutative domain with finitely generated fraction field. So we should at least adopt the locally algebraic assumption. Unfortunately, it is difficult to apply the result of [29] to a general GWA; the process of inverting z, as in Corollary 23, does not make it simple to carry along GK dimension information. For one thing, z is typically not central or even normal in W. Also, a locally algebraic σ can fail to induce a locally algebraic automorphism of the localized algebra. So we instead proceed with a direct calculation:

Theorem 28: *Assume that the automorphism $\sigma : R \to R$ is locally algebraic. Then*

$$\mathrm{GK}(W) = \mathrm{GK}(R) + 1.$$

Proof: Given Proposition 26, it remains to show that $\mathrm{GK}(W) \leq \mathrm{GK}(R) + 1$. Let Z denote the linear span of $\{\sigma^i(z) \mid i \in \mathbb{Z}\} \cup \{1\}$. Consider any affine subalgebra of W; let X be a finite dimensional generating subspace for it. We first enlarge X to a subspace $\bar{X}$ of the form

$$\bar{X} := \bigoplus_{|m|\le m_0} U\, v_m, \tag{15}$$

where U is a finite dimensional σ-stable subspace of R with $Z \subseteq U$. Here is a procedure for doing this: for $m \in \mathbb{Z}$, let $\pi_m : W \to R$ denote the m^{th} projection map coming from the left R-basis $(v_m)_{m\in\mathbb{Z}}$ of W. Let $m_0 = \max\{|m| \mid \pi_m(X) \neq 0\}$. Now $\sum_{|m|\le m_0} \pi_m(X)$ is a finite dimensional subspace of R, so it is contained in a finite dimensional σ-stable subspace U of R. It is harmless to include Z in U (note that Z is finite dimensional because σ, and hence also σ^{-1}, is locally algebraic). This gives us $\bar{X}$ defined by (15), with $X \subseteq \bar{X}$.

Next, we show that

$$\bar{X}^n \subseteq \bigoplus_{|m|\le nm_0} U^{n+(n-1)m_0} v_m \tag{16}$$

for $n \geq 1$. It holds by definition when $n = 1$, so assume that $n > 1$ and that (16) holds for $\bar{X}^{n-1}$. Then the induction goes through:

$$\begin{aligned}
\bar{X}^n &= \bar{X}^{n-1}\bar{X} \subseteq \left(\bigoplus_{|m|\le(n-1)m_0} U^{n-1+(n-2)m_0} v_m\right)\left(\bigoplus_{|m|\le m_0} U\, v_m\right)\\
&= \bigoplus_{|m|\le nm_0} \sum_{\substack{m_1+m_2=m\\ |m_1|\le(n-1)m_0\\ |m_2|\le m_0}} U^{n-1+(n-2)m_0} v_{m_1}\, U\, v_{m_2}\\
&\subseteq \bigoplus_{|m|\le nm_0} \sum_{\substack{m_1+m_2=m\\ |m_1|\le(n-1)m_0\\ |m_2|\le m_0}} U^{n+(n-1)m_0} v_{m_1+m_2} = \bigoplus_{|m|\le nm_0} U^{n+(n-1)m_0} v_m.
\end{aligned}$$

For the inclusion in the final line we used the fact, evident from (2), that $[\![m_1, m_2]\!] \in Z^{\min(|m_1|,|m_2|)} \subseteq U^{\min(|m_1|,|m_2|)}$. With (16) established, we have

$$\dim(\bar{X}^n) \le (2nm_0 + 1)\dim(U^{n+(n-1)m_0})$$

for all $n \geq 1$. The theorem follows:

$$\mathrm{GK}(k\langle X\rangle) \le \mathrm{GK}(k\langle \bar{X}\rangle) \le 1 + \mathrm{GK}(R). \qquad \blacksquare$$

2.5 Representation Theory

Modules over GWAs have been explored and classified under various hypotheses by several authors. A classification of simple $R[x, y; \sigma, z]$-modules is obtained in [5] for R a Dedekind domain with restricted minimum condition and with a condition placed on σ: that maximal ideals of R are never fixed by any nonzero power of σ. These results are expanded in [7] and further in [14], where indecomposable weight modules with finite length as R-modules are classified for R commutative. In the latter work, the authors introduce *chain* and *circle* categories to handle maximal ideals of R that have infinite and finite σ-orbit respectively. Another expansion of the work of [5] was carried out in [35], where the simple R-torsion modules were classified relaxing all assumptions on R (even commutativity), but with the assumption that σ acts freely on the set of maximal left ideals of R. In order to establish notation and put the spotlight on a particular setting that will be of use to us, we proceed with our own development.

2.5.1 Simple Modules

Let R be a commutative k-algebra and let $W = R[x, y; \sigma, z]$ be a GWA. Let ${}_W V$ be a finite dimensional simple left W-module. It contains some simple left R-module V_0, which has an annihilator $\mathfrak{m} := \operatorname{ann}_R V_0 \in \operatorname{max\,spec} R$. The automorphism σ acts on $\operatorname{max\,spec} R$, and the behavior of V depends largely on whether $\mathfrak{m}$ sits in a finite or an infinite orbit. We would like to deal with the infinite orbit case, so assume that $\sigma^i(\mathfrak{m}) = \sigma^j(\mathfrak{m}) \Rightarrow i = j$ for $i, j \in \mathbb{Z}$.

Let e_0 be a nonzero element of V_0, so we have $\mathfrak{m} = \operatorname{ann}_R e_0$. For $i \in \mathbb{Z}$, let $e_i = v_i.e_0$. Notice that for $i \in \mathbb{Z}$ and $r \in \mathfrak{m}$, we have

$$\sigma^i(r).e_i = \sigma^i(r)v_i.e_0 = v_i r.e_0 = 0,$$

so $\sigma^i(\mathfrak{m}) \subseteq \operatorname{ann}_R e_i$. So whenever $e_i \neq 0$, $\sigma^i(\mathfrak{m}) = \operatorname{ann}_R e_i$. We use this to argue that the subspaces Re_i are independent: Consider a vanishing combination

$$\sum_{i \in \mathscr{I}} r_i e_i = 0 \tag{17}$$

where $\mathscr{I} \subseteq \mathbb{Z}$ is finite and $e_i \neq 0$ for $i \in \mathscr{I}$. For any $j \in \mathscr{I}$, choose a $c_j \in \left(\prod_{i \in \mathscr{I} \setminus \{j\}} \sigma^i(\mathfrak{m})\right) \setminus \sigma^j(\mathfrak{m})$, and apply it to (17). The result is $c_j r_j e_j = 0$, which implies that $c_j r_j \in \sigma^j(\mathfrak{m})$, so $r_j \in \sigma^j(\mathfrak{m})$ and $r_j e_j = 0$.

Since we assumed V to be finite dimensional, only finitely many of the e_i may be nonzero. In particular, there is some $e_{i_0} \neq 0$ such that $e_{i_0 - 1} = 0$ (a "lowest weight vector"). We may as well shift our original indexing so that this e_{i_0} is e_0. (After all, e_0 was only assumed to be a nonzero element of some simple R-submodule of V with annihilator having infinite σ-orbit, and e_{i_0} would have fit the bill just as well.) Similarly, on the other end, there is some $n \geq 0$ so that $e_{n-1} \neq 0$ and $e_n = 0$. Note that these definitions imply that $e_i = x^i.e_0$ is nonzero for $0 \leq i \leq n-1$.

It is now clear that $\bigoplus_{i=0}^{n-1} Re_i$ is a W-submodule of V:

$$\begin{aligned} x(re_i) &= \sigma(r)xe_i &&= \sigma(r)e_{i+1} \\ y(re_i) &= \sigma^{-1}(r)ye_i &&= \sigma^{-1}(r)ze_{i-1}. \end{aligned} \tag{18}$$

So, since ${}_W V$ is simple, $\bigoplus_{i=0}^{n-1} Re_i = V$. Each Re_i for $0 \leq i \leq n-1$ is isomorphic as an R-module to $R/\sigma^i(\mathfrak{m})$. Knowing this and knowing that the W-action is described by (18), we have pinned down ${}_W V$ up to isomorphism. Let us also pin down e_0 and $\mathfrak{m}$.

Applying xy and yx to the extreme "edges" of the module shows that $\sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m}$:

$$\sigma(z).e_0 = x.(y.e_0) = 0 \;\Rightarrow\; \sigma(z) \in \mathfrak{m}$$
$$z.e_{n-1} = y.(x.e_{n-1}) = 0 \;\Rightarrow\; z \in \sigma^{n-1}(\mathfrak{m}).$$

Further, $n > 0$ is *minimal* with respect to this property: if we had $0 < i < n$ with $\sigma^{-i+1}(z) \in \mathfrak{m}$, then $y.e_i = 0$, so $Re_i + \cdots + Re_n$ would be a proper nontrivial submodule of V.

The following definition will be useful throughout this work.

Definition 29: Given a commutative ring R, an automorphism σ, an element z, and a maximal ideal $\mathfrak{m}$, define:

$$n(\mathfrak{m}, \sigma, z) = \min\{n \in \mathbb{Z} \mid n > 0 \text{ and } \sigma^{-n+1}(z) \in \mathfrak{m}\}$$
$$n'(\mathfrak{m}, \sigma, z) = \min\{n' \in \mathbb{Z} \mid n' > 0 \text{ and } \sigma^{n'}(z) \in \mathfrak{m}\},$$

where $\min(\emptyset)$ is taken to be ∞. When the context is clear, we drop some notation and simply write $n(\mathfrak{m})$ or $n'(\mathfrak{m})$.

We may now characterize Re_0 as $\mathrm{ann}_V(y)$, as follows. The inclusion $Re_0 \subseteq \mathrm{ann}_V(y)$ is obvious since y normalizes R. Suppose that $y.\left(\sum_{i=0}^{n-1} r_i e_i\right) = 0$, where $r_i \in R$. Then $0 = \sum_{i=1}^{n-1} \sigma^{-1}(r_i) z e_{i-1}$, so for each $1 \leq i \leq n-1$ we have $\sigma^{-1}(r_i)z \in \sigma^{i-1}(\mathfrak{m})$. The minimality of n discussed above implies that $z \notin \sigma^{i-1}(\mathfrak{m})$, so we have $\sigma^{-1}(r_i) \in \sigma^{i-1}(\mathfrak{m})$, and hence $r_i \in \sigma^i(\mathfrak{m}) = \mathrm{ann}_R(e_i)$, for $1 \leq i \leq n-1$. So $\sum_{i=0}^{n-1} r_i e_i = r_0 e_0 \in Re_0$, proving that $\mathrm{ann}_V(y) = Re_0$. We have also gained a nice internal description for $\mathfrak{m}$: it is $\mathrm{ann}_R(\mathrm{ann}_V(y))$. Let us record what has been established so far:

Lemma 30: *Let ${}_W V$ be a finite dimensional simple left W-module, where $W = R[x, y; \sigma, z]$ and R is a commutative k-algebra. Assume that V contains some simple R-submodule with annihilator having infinite σ-orbit. Then $\mathrm{ann}_V(y)$ is just such an R-submodule. Let $\mathfrak{m} = \mathrm{ann}_R(\mathrm{ann}_V(y))$. Then $n'(\mathfrak{m}) = 1$, we have that $n := n(\mathfrak{m})$ is finite,and V is isomorphic to*

$$\bigoplus_{i=0}^{n-1} R/\sigma^i(\mathfrak{m}) \tag{19}$$

as an R-module. Let e_i denote $1 \in R/\sigma^i(\mathfrak{m})$ as an element of (19) for $0 \leq i \leq n-1$, and let $e_{-1} = e_n = 0$. Then ${}_W V$ is isomorphic to (19) if (19) is given the following W-action:

$$x(re_i) = \sigma(r)e_{i+1}$$
$$y(re_i) = \sigma^{-1}(r)ze_{i-1}.$$

One could check explicitly that forming the R-module (19) and defining actions of x and y according to (18) yields a well-defined, simple, and finite-dimensional module over W. But we can learn a bit more about W by instead realizing these modules as quotients by certain left ideals. We will run into a family of infinite dimensional simple modules along the way; the construction mimics the Verma modules typical to the treatment of representations of $\mathfrak{sl}_2$ [22, II.7] and $U_q(\mathfrak{sl}_2)$ [8, I.4].

Definition 31: Let R be a commutative ring, $W = R[x, y; \sigma, z]$, and $\mathfrak{m}$ a maximal ideal of R with infinite σ-orbit. Define $I_\mathfrak{m} := W\mathfrak{m}$ to be the left ideal of W that $\mathfrak{m}$ generates, and define $M_\mathfrak{m}$ to be the $\mathbb{Z}$-graded left W-module $M_\mathfrak{m} := W/I_\mathfrak{m}$. Define e_i to be the image of v_i in $M_\mathfrak{m}$ for $i \in \mathbb{Z}$.

Note that

$$I_\mathfrak{m} = \bigoplus_{i \in \mathbb{Z}} \sigma^i(\mathfrak{m}) v_i;$$

the inclusion $\supseteq$ is due to the fact that $v_i \mathfrak{m} = \sigma^i(\mathfrak{m}) v_i$, and $\subseteq$ holds because the right hand side is a left ideal of W (condition (8) is satisfied).

Lemma 32: *Let R be a commutative k-algebra, $W = R[x, y; \sigma, z]$, and $\mathfrak{m}$ a maximal ideal of R with infinite σ-orbit. The submodules of $M_\mathfrak{m}$ are of the following types:*

1. *0 or $M_\mathfrak{m}$*
2. *$\bigoplus_{i \geq j} Re_i$ for some $j > 0$ with $\sigma^{-j+1}(z) \in \mathfrak{m}$*
3. *$\bigoplus_{i \leq -j'} Re_i$ for some $j' > 0$ with $\sigma^{j'}(z) \in \mathfrak{m}$*
4. *a sum of a submodule of type 2 and one of type 3.*

Proof: Let S be a proper nontrivial submodule of $M_\mathfrak{m}$. We first show that S is homogeneous, so that if $\sum a_i e_i \in S$ with a certain $a_j e_j \neq 0$, then $e_j \in S$.

> **Claim:** S is homogeneous.
> **Proof:** Suppose that $a \in S$, say $a = \sum_{i \in \mathscr{I}} a_i e_i$ with $\mathscr{I} \subseteq \mathbb{Z}$ finite and $a_i \in R \setminus \sigma^i(\mathfrak{m})$ for $i \in \mathscr{I}$. Let $j \in \mathscr{I}$, and choose an element c of $\left(\prod_{i \in \mathscr{I} \setminus \{j\}} \sigma^i(\mathfrak{m})\right) \setminus \sigma^j(\mathfrak{m})$. Then $ca = ca_j e_j \in S$. Since $c, a_j \in R \setminus \sigma^j(\mathfrak{m})$, ca_j is a unit mod $\sigma^j(\mathfrak{m})$. Hence $e_j \in S$.

Define vector subspaces $M^+ := \bigoplus_{i>0} Re_i$ and $M^- := \bigoplus_{i<0} Re_i$ of $M_\mathfrak{m}$. Since S is proper and homogeneous,

$$S = (S \cap M^+) \oplus (S \cap M^-).$$

To show that S is of type 2, 3, or 4, then, it suffices to show that $S \cap M^+$ is a type 2 submodule when it is nonzero, and that $S \cap M^-$ is a type 3 submodule when it is nonzero.

Assume that $S \cap M^+ \neq 0$. Then $e_j \in S$ for some $j > 0$; let $j > 0$ be minimal such that this happens. By applying powers of x, we see that $S \cap M^+ = \bigoplus_{i \geq j} Re_i$. Since $e_{j-1} \notin S$, $ye_j = ze_{j-1}$ must vanish. This happens if and only if $z \in \sigma^{j-1}(\mathfrak{m})$, i.e. if and only if

$$\sigma^{-j+1}(z) \in \mathfrak{m}. \tag{20}$$

Now assume that $S \cap M^- \neq 0$. Let $j' > 0$ be minimal such that $e_{-j'} \in S$. By applying powers of y, we see that $S \cap M^- = \bigoplus_{i \leq -j'} Re_i$. Since $e_{-j'+1} \notin S$, $xe_{-j'} = \sigma(z)e_{-j'+1}$ must vanish. This happens if and only if $\sigma(z) \in \sigma^{-j'+1}(\mathfrak{m})$, i.e. if and only if

$$\sigma^{j'}(z) \in \mathfrak{m}. \tag{21}$$

Finally, it is routine to check that 1-4 are actually submodules of $M_\mathfrak{m}$, considering the equivalences mentioned in (20) and (21). ■

This shows that $M_\mathfrak{m}$ has a unique largest proper submodule, $N_\mathfrak{m}$, given by

$$N_\mathfrak{m} := \bigoplus_{i \leq -n'(\mathfrak{m})} Re_i \ \oplus \bigoplus_{i \geq n(\mathfrak{m})} Re_i. \tag{22}$$

Note that the $n(\mathfrak{m}), n'(\mathfrak{m})$ in the above expression could be ∞. For example, if $\mathfrak{m}$ is disjoint from $\{\sigma^i(z) \mid i \in \mathbb{Z}\}$, then $N_\mathfrak{m} = 0$ and $M_\mathfrak{m}$ is simple.

Throughout this work, it will be useful to have notation for certain subsets of $\max\operatorname{spec}(R)$:

Definition 33: Given a commutative ring R, an automorphism σ, and an element z, define:

$$\begin{aligned}
\mathscr{M}(R,\sigma) &= \{\mathfrak{m} \in \max\operatorname{spec} R \mid \mathfrak{m} \text{ has infinite } \sigma\text{-orbit}\} \\
\mathscr{M}_{\mathrm{I}}(R,\sigma,z) &= \{\mathfrak{m} \in \mathscr{M} \mid n(\mathfrak{m},\sigma,z) = \infty \text{ or } n'(\mathfrak{m},\sigma,z) = \infty\} \\
\mathscr{M}_{\mathrm{II}}(R,\sigma,z) &= \{\mathfrak{m} \in \mathscr{M} \mid n(\mathfrak{m},\sigma,z), n'(\mathfrak{m},\sigma,z) < \infty\} \\
\mathscr{M}'_{\mathrm{II}}(R,\sigma,z) &= \{\mathfrak{m} \in \mathscr{M} \mid n'(\mathfrak{m},\sigma,z) = 1 \text{ and } n(\mathfrak{m},\sigma,z) < \infty\}.
\end{aligned}$$

When the context is clear, we will drop some notation and simply write $\mathscr{M}$, $\mathscr{M}_{\mathrm{I}}$, $\mathscr{M}_{\mathrm{II}}$, and $\mathscr{M}'_{\mathrm{II}}$.

We now state the classification theorem for simple modules.

Theorem 34: *Let R be a commutative k-algebra and $W = R[x,y;\sigma,z]$.*

1. *Let $\mathfrak{m} \in \mathscr{M}$. Assume that R is affine. The simple module $V_\mathfrak{m} := M_\mathfrak{m}/N_\mathfrak{m}$ is finite dimensional if and only if $\mathfrak{m} \in \mathscr{M}_{II}$.*
2. *Any finite dimensional simple left W-module V that contains a simple R-submodule with annihilator having infinite σ-orbit is isomorphic to $V_\mathfrak{m}$ for exactly one $\mathfrak{m} \in \mathscr{M}'_{II}$, namely $\mathfrak{m} = \operatorname{ann}_R(\operatorname{ann}_V(y))$.*
3. *Let $\mathfrak{m} \in \mathscr{M}'_{II}$ and let $n := n(\mathfrak{m})$. Then $V_\mathfrak{m} \cong W/(W\mathfrak{m} + Wy + Wx^n)$.*

Proof: Assertion 1 follows from Lemma 32, the definition of $N_\mathfrak{m}$, and the fact (due to the Nullstellensatz) that each $R/\sigma^i(\mathfrak{m})$ is finite dimensional when R is affine. For assertion 2, suppose that $_WV$ is simple, finite dimensional, and contains a simple R-submodule with annihilator having infinite σ-orbit. Lemma

30 pins V down as isomorphic to the left W-module in (19). This construction is in turn isomorphic to $V_{\mathfrak{m}}$, where $\mathfrak{m} = \operatorname{ann}_R(\operatorname{ann}_V(y))$, and the lemma guarantees that $\sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m}$ for some $n > 0$. Hence $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$ and $V \cong V_{\mathfrak{m}}$. Since $\mathfrak{m} = \operatorname{ann}_R(\operatorname{ann}_{V_{\mathfrak{m}}}(y))$, no two $V_{\mathfrak{m}}$ for $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$ can be isomorphic. Assertion 3 amounts to the fact that, under the given hypotheses, $N_{\mathfrak{m}}$ is the submodule of $M_{\mathfrak{m}}$ generated by the cosets $y + I_{\mathfrak{m}}$ and $x^n + I_{\mathfrak{m}}$. ■

Given a commutative ring R, the *support* $\operatorname{Supp} X$ of an R-module X is defined to be the collection of maximal ideals $\mathfrak{m}$ of R such that $\operatorname{ann}_X \mathfrak{m}$ is nonzero. Before ending this section, we take a moment to record an explicit description of the support of a $V_{\mathfrak{m}}$-type module. This will be useful in section 2.6, where $V_{\mathfrak{m}}$ will make another appearance.

Proposition 35: *Let $W = R[x, y; \sigma, z]$ be a GWA over a commutative k-algebra R. For any $\mathfrak{m} \in \max\operatorname{spec}(R)$, we have $\operatorname{Supp}({}_R M_{\mathfrak{m}}) = \{\sigma^j(\mathfrak{m}) \mid j \in \mathbb{Z}\}$.*

Proof: Fix $j \in \mathbb{Z}$. Proposition 4 tells us that W is a free right R-module on the basis $(v_m)_{m\in\mathbb{Z}}$. It follows that $v_j \notin v_j\mathfrak{m}$. Hence the image of v_j in

$$(M_{\mathfrak{m}})_R \cong \bigoplus_{m\in\mathbb{Z}} v_m R/(v_m\mathfrak{m})$$

is nonzero. Since the image of v_j in $M_{\mathfrak{m}}$ lies in the component $Rv_j/(Rv_j\mathfrak{m})$ (which is $Rv_j/(\sigma^j(\mathfrak{m})v_j)$), it is annihilated by $\sigma^j(\mathfrak{m})$ on the left. This shows that $\{\sigma^j(\mathfrak{m}) \mid j \in \mathbb{Z}\} \subseteq \operatorname{Supp}({}_R M_{\mathfrak{m}})$. The reverse inclusion is clear. ■

Proposition 36: *Let $W = R[x, y; \sigma, z]$ be a GWA over a commutative k-algebra R. For any $\mathfrak{m} \in \mathscr{M}$,*

$$\operatorname{Supp}({}_R V_{\mathfrak{m}}) = \{\sigma^j(\mathfrak{m}) \mid -n'(\mathfrak{m}) < j < n(\mathfrak{m})\}. \tag{23}$$

Proof: This follows from the expression (22) for $N_{\mathfrak{m}}$ and Proposition 35. ■

2.5.2 Weight Modules

In further pursuit of finite dimensional modules, we now explore a class of modules that includes the semisimple ones. We continue with the notation $W = R[x, y; \sigma, z]$ and the assumption that R is a commutative k-algebra. Let ${}_W X$ be finite dimensional and semisimple. Consider the R-submodule spanned by annihilators of maximal ideals,

$$S := \sum_{\mathfrak{m}\in\max\operatorname{spec} R} \operatorname{ann}_X \mathfrak{m}.$$

It is in fact a W-submodule of X, since x and y map $\operatorname{ann}_X \mathfrak{m}$ into $\operatorname{ann}_X \sigma(\mathfrak{m})$ and $\operatorname{ann}_X \sigma^{-1}(\mathfrak{m})$ respectively. Since we assumed X to be semisimple, S has a direct sum complement S' in ${}_W X$. If S' were nonzero, then it contains some simple R-submodule which is then annihilated by some maximal ideal of R, contradicting $S' \cap S = 0$. Thus our assumption that ${}_W X$ is semisimple requires $X = S$. We now wonder when this condition is sufficient for semisimplicity.

Definition 37: Let R be a commutative k-algebra. A W-module where $W = R[x, y; \sigma, z]$ is a *weight* module if and only if it is semisimple as an R-module. Note that this is equivalent to saying that X is spanned by annihilators of maximal ideals of R.

Let us collect some elementary facts about weight modules for use in the coming semisimplicity theorem.

Proposition 38: *Let R be a commutative k-algebra, X a semisimple R-module, and ${}_RY \leq {}_RX$. Then*

$$X = \bigoplus_{\mathfrak{m} \in \max \operatorname{spec} R} \operatorname{ann}_X \mathfrak{m}$$

and the canonical map $X \to X/Y$ induces an isomorphism of R-modules

$$(\operatorname{ann}_X \mathfrak{m})/(\operatorname{ann}_Y \mathfrak{m}) \cong \operatorname{ann}_{X/Y} \mathfrak{m}$$

for each $\mathfrak{m} \in \max \operatorname{spec} R$.

Proof: By assumption, X is a direct sum of simple R-submodules. Each simple R-submodule is isomorphic to $R/\mathfrak{m}$ for some $\mathfrak{m} \in \max \operatorname{spec} R$. Thus X is a direct sum of the $\operatorname{ann}_X \mathfrak{m}$; each $\operatorname{ann}_X \mathfrak{m}$ is actually just the $(R/\mathfrak{m})$-homogeneous component of X. Since Y is a submodule of X, it is semisimple and has its own decomposition

$$Y = \bigoplus_{\mathfrak{m} \in \max \operatorname{spec} R} \operatorname{ann}_Y \mathfrak{m} = \bigoplus_{\mathfrak{m} \in \max \operatorname{spec} R} Y \cap \operatorname{ann}_X \mathfrak{m}. \tag{24}$$

Fix an $\mathfrak{m} \in \max \operatorname{spec} R$. It is clear that the canonical map $X \to X/Y$ restricts to an R-homomorphism $\operatorname{ann}_X \mathfrak{m} \to \operatorname{ann}_{X/Y} \mathfrak{m}$ with kernel $Y \cap \operatorname{ann}_X \mathfrak{m} = \operatorname{ann}_Y \mathfrak{m}$. To see that it is surjective, consider any $x + Y \in \operatorname{ann}_{X/Y} \mathfrak{m}$. Write x as $\sum_{\mathfrak{n} \in \max \operatorname{spec} R} x_\mathfrak{n}$, where $x_\mathfrak{n} \in \operatorname{ann}_X \mathfrak{n}$. Since $\mathfrak{m}x \subseteq Y$, the decomposition (24) gives $\mathfrak{m}x_\mathfrak{n} \subseteq Y$ for all $\mathfrak{n}$. When $\mathfrak{n} \neq \mathfrak{m}$, this implies that $x_\mathfrak{n} \in Y$ since $\mathfrak{m}$ contains a unit mod $\mathfrak{n}$. Thus $x + Y$ is the image of $x_\mathfrak{m}$ under $\operatorname{ann}_X \mathfrak{m} \to \operatorname{ann}_{X/Y} \mathfrak{m}$. ■

Since we only focused on simple finite dimensional W-modules of a certain type, we will only attempt to get at the weight modules whose composition factors are of that type. Adapting the "chain" and "circle" terminology from [14]:

Definition 39: Let R be a commutative k-algebra and let $W = R[x, y; \sigma, z]$. A finite dimensional module ${}_WX$ is of *chain-type* if and only if $\operatorname{Supp} X \subseteq \mathscr{M}$.

Proposition 40: *Let R be a commutative k-algebra, $W = R[x, y; \sigma, z]$, and ${}_WX$ a chain-type finite dimensional weight module. Then each composition factor of X has the form $V_\mathfrak{m}$ for some $\mathfrak{m} \in \mathscr{M}'_{II}$, and*

$$\operatorname{Supp} X \cap \mathscr{M}'_{II} = \{\mathfrak{m} \in \mathscr{M}'_{II} \mid V_\mathfrak{m} \textit{ is a composition factor of } {}_WX\}. \tag{25}$$

Proof: Choose a W-module composition series $0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X$. It can be refined into a composition series for ${}_RX$, so since ${}_RX$ is semisimple we have:

$${}_RX \cong \bigoplus_{i=1}^{r} \bigoplus \{(R/\mathfrak{m})^{(k)} \mid R/\mathfrak{m} \text{ is a composition factor of } X_i/X_{i-1} \text{ with multiplicity } k\}. \tag{26}$$

In particular, each X_i/X_{i-1} contains some simple R-submodule whose annihilator comes from $\operatorname{Supp} X$ and therefore has infinite σ-orbit. Theorem 34 applies: for $1 \leq i \leq r$, $X_i/X_{i-1} \cong V_{\mathfrak{m}_i}$ for a unique $\mathfrak{m}_i \in \mathscr{M}'_{\mathrm{II}}$. The right hand side of (25) is then $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$.

Knowing that $X_i/X_{i-1} \cong V_{\mathfrak{m}_i}$, we can read off the support of X from (26):

$$\operatorname{Supp} X = \{\sigma^\ell(\mathfrak{m}_i) \mid 1 \leq i \leq r \text{ and } 0 \leq \ell \leq n(\mathfrak{m}_i) - 1\}.$$

Suppose that $\sigma^\ell(\mathfrak{m}_i) \in \mathscr{M}'_{\mathrm{II}}$, with $1 \leq i \leq r$ and $0 \leq \ell \leq n(\mathfrak{m}_i) - 1$. Then $\sigma(z) \in \sigma^\ell(\mathfrak{m}_i)$, so $\sigma^{-\ell+1}(z) \in \mathfrak{m}_i$. The minimality of $n(\mathfrak{m}_i)$ forces $\ell = 0$. This proves that $\operatorname{Supp} X \cap \mathscr{M}'_{\mathrm{II}} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$, and the latter is the right hand side of (25). ■

Next, we identify a condition on $\operatorname{Supp} X \cap \mathscr{M}'_{\mathrm{II}}$ that we will show guarantees semisimplicity for ${}_WX$.

Definition 41: Let R be a commutative k-algebra, σ an automorphism, and $z \in R$. A subset $S \subset \mathscr{M}'_{\mathrm{II}}$ *has separated chains* if $\mathfrak{m} \in S \Rightarrow \sigma^{n(\mathfrak{m})}(\mathfrak{m}) \notin S$ for all $\mathfrak{m}$.

Proposition 42: *Let R be a commutative k-algebra, σ an automorphism, and $z \in R$. Assume that $S \subset \mathscr{M}'_{II}$ has separated chains. Then given $\mathfrak{m}, \mathfrak{m}_0 \in S$ and letting $n := n(\mathfrak{m})$ and $n_0 := n(\mathfrak{m}_0)$,*

1. $\mathfrak{m}_0 \in \{\mathfrak{m}, \sigma(\mathfrak{m}), \dots, \sigma^{n-1}(\mathfrak{m})\}$ *only if* $\mathfrak{m}_0 = \mathfrak{m}$
2. $\sigma^{-1}(\mathfrak{m}_0), \sigma^{n_0}(\mathfrak{m}_0) \notin \{\mathfrak{m}, \sigma(\mathfrak{m}), \dots, \sigma^{n-1}(\mathfrak{m})\}$.

Proof: Suppose that $\mathfrak{m}_0 = \sigma^{\ell}(\mathfrak{m})$, where $0 \leq \ell \leq n-1$. Then $\sigma(z) \in \sigma^{\ell}(\mathfrak{m})$, so $\sigma^{-\ell+1}(z) \in \mathfrak{m}$. The minimality of n then forces $\ell = 0$, whence $\mathfrak{m}_0 = \mathfrak{m}$.

Suppose that $\sigma^{-1}(\mathfrak{m}_0) = \sigma^{\ell}(\mathfrak{m})$, where $0 \leq \ell \leq n-1$. Then $\sigma(z) \in \mathfrak{m}_0 = \sigma^{\ell+1}(\mathfrak{m})$, so $\sigma^{-(\ell+1)+1}(z) \in \mathfrak{m}$. The minimality of n then forces $\ell+1 = n$, which gives $\sigma^{n}(\mathfrak{m}) = \mathfrak{m}_0 \in S$. This contradicts the assumption that S has separated chains.

Suppose that $\sigma^{n_0}(\mathfrak{m}_0) = \sigma^{\ell}(\mathfrak{m})$, where $0 \leq \ell \leq n-1$. Then we have $\sigma^{-\ell+1}(z) \in \mathfrak{m}$, since $\sigma^{-n_0+1}(z) \in \mathfrak{m}_0$. The minimality of n then forces $\ell = 0$, which gives $\sigma^{n_0}(\mathfrak{m}_0) = \mathfrak{m} \in S$, contradicting the assumption that S has separated chains. ∎

Theorem 43: *Let R be a commutative k-algebra, let $W = R[x, y; \sigma, z]$, and let X be a chain-type finite dimensional weight left W-module. If $\operatorname{Supp} X \cap \mathscr{M}'_{II}$ has separated chains, then X is semisimple.*

Proof: Assume the hypotheses. Choose a composition series for ${}_W X$:

$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X.$$

For $1 \leq i \leq r$, $X_i/X_{i-1} \cong V_{\mathfrak{m}_i}$ for a unique $\mathfrak{m}_i \in \mathscr{M}'_{\mathrm{II}}$, and $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ has separated chains (Proposition 40). Let $\mathfrak{n}_1, \dots, \mathfrak{n}_s$ be the distinct items among $\mathfrak{m}_1, \dots, \mathfrak{m}_r$, with respective multiplicities $t_1, \dots, t_s$. For each $1 \leq j \leq s$, let $n_j := n(\mathfrak{n}_j)$.

For any $\mathfrak{a} \in \max\operatorname{spec} R$, we iteratively apply Proposition 38 to obtain:

$$\begin{aligned} \dim_{R/\mathfrak{a}} \operatorname{ann}_X \mathfrak{a} &= \dim_{R/\mathfrak{a}} \operatorname{ann}_{X_r/X_{r-1}} \mathfrak{a} + \dim_{R/\mathfrak{a}} \operatorname{ann}_{X_{r-1}} \mathfrak{a} = \cdots = \sum_{i=1}^{r} \dim_{R/\mathfrak{a}} \operatorname{ann}_{X_i/X_{i-1}} \mathfrak{a} \\ &= \sum_{i=1}^{r} \dim_{R/\mathfrak{a}} \operatorname{ann}_{V_{\mathfrak{m}_i}} \mathfrak{a} \end{aligned} \tag{27}$$

Fix a j with $1 \leq j \leq s$. Apply (27) to the case $\mathfrak{a} = \mathfrak{n}_j$ and use Proposition 42.1 to obtain

$$\dim_{R/\mathfrak{n}_j} \operatorname{ann}_X \mathfrak{n}_j = t_j.$$

Apply (27) to the cases $\mathfrak{a} = \sigma^{-1}(\mathfrak{n}_j)$ and $\mathfrak{a} = \sigma^{n_j}(\mathfrak{n}_j)$ and use Proposition 42.2 to obtain

$$\operatorname{ann}_X(\sigma^{-1}(\mathfrak{n}_j)) = \operatorname{ann}_X(\sigma^{n_j}(\mathfrak{n}_j)) = 0.$$

Let $b_1^j, \dots, b_{t_j}^j$ be an $(R/\mathfrak{n}_j)$-basis for $\operatorname{ann}_X \mathfrak{n}_j$. Each $W b_u^j$ is a nonzero homomorphic image of ${}_W W$ in which $\mathfrak{n}_j$, y, and x^{n_j} are killed: $\mathfrak{n}_j$ is killed because b_u^j came from $\operatorname{ann}_X \mathfrak{n}_j$, and y and x^{n_j} are killed because they map $\operatorname{ann}_X \mathfrak{n}_j$ into $\operatorname{ann}_X(\sigma^{-1}(\mathfrak{n}_j))$ and $\operatorname{ann}_X(\sigma^{n_j}(\mathfrak{n}_j))$ respectively. By Theorem 34.3, it follows that each $W b_u^j$ is isomorphic to $M_{\mathfrak{n}_j}/N_{\mathfrak{n}_j} =: V_{\mathfrak{n}_j}$.

Do this for all j. Let

$$S = \sum_{j=1}^{s} \sum_{u=1}^{t_j} W b_u^j,$$

a semisimple W-submodule of X. Since $\operatorname{ann}_X \mathfrak{n}_j$ is R-spanned by $b_1^j, \ldots, b_{t_j}^j$, we have (by Proposition 38)

$$\operatorname{ann}_{X/S} \mathfrak{n}_j \cong (\operatorname{ann}_X \mathfrak{n}_j)/(\operatorname{ann}_S \mathfrak{n}_j) = 0$$

for all j. By the Jordan-Hölder theorem, any simple W-submodule of X/S is isomorphic to $V_{\mathfrak{n}_j}$ for some j. Therefore X/S must be 0. That is, $X = S$ is semisimple. ■

If $\mathscr{M}'_{\mathrm{II}}$ as a whole has separated chains, then we conclude from this theorem that *all* chain-type finite dimensional weight modules are semisimple. There is a converse:

Proposition 44: *Let R be an affine commutative k-algebra and let $W = R[x, y; \sigma, z]$. If $\mathscr{M}'_{II}$ does not have separated chains, then there is a chain-type finite dimensional weight left W-module that is not semisimple.*

Proof: If $\mathscr{M}'_{\mathrm{II}}$ does not have separated chains, there is some $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$ such that $\sigma^{n(\mathfrak{m})}(\mathfrak{m}) \in \mathscr{M}'_{\mathrm{II}}$. Let $n := n(\mathfrak{m})$ and let $n_1 = n(\sigma^n(\mathfrak{m}))$. Then $\mathfrak{m}$ contains $\sigma(z)$, $\sigma^{-n+1}(z)$, and $\sigma^{-(n+n_1)+1}(z)$. Hence

$$S := \left(\bigoplus_{i \leq -1} Re_i \right) \oplus \left(\bigoplus_{i \geq n+n_1} Re_i \right)$$

is a submodule of $M_{\mathfrak{m}}$, by Lemma 32. Let ${}_W X = M_{\mathfrak{m}}/S$. This is isomorphic to $\bigoplus_{0 \leq i < n+n_1} R/\sigma^i(\mathfrak{m})$ as an R-module, so ${}_W X$ is a chain-type finite dimensional weight left W-module. Since $M_{\mathfrak{m}}$ contains a unique largest proper submodule

$$N_{\mathfrak{m}} = \left(\bigoplus_{i \leq -1} Re_i \right) \oplus \left(\bigoplus_{i \geq n} Re_i \right)$$

and $N_{\mathfrak{m}}$ properly contains S, X contains a unique largest proper nontrivial submodule $N_{\mathfrak{m}}/S$. Therefore X cannot be semisimple. ■

Theorem 45: *Let R be an affine commutative k-algebra and let $W = R[x, y; \sigma, z]$. The following are equivalent:*

1. *All chain-type finite dimensional weight left W-modules are semisimple.*
2. *$\mathscr{M}'_{II}$ has separated chains.*
3. *For any $\mathfrak{m} \in \mathscr{M}$, there are no more than two integers i such that $\sigma^i(z) \in \mathfrak{m}$.*
4. *For any $\mathfrak{m} \in \mathscr{M}'_{II}$, there is exactly one $n > 0$ such that $\sigma^{-n+1}(z) \in \mathfrak{m}$.*

Proof: The equivalence between 1 and 2 is due to Theorem 43 and Proposition 44.

$2 \Rightarrow 4$: Assume that 4 fails. Let $\mathfrak{m}$ be in $\mathscr{M}'_{\mathrm{II}}$ and let $i < j$ be positive integers such that $\sigma^{-i+1}(z), \sigma^{-j+1}(z) \in \mathfrak{m}$. We may assume that $i > 0$ is minimal such that $\sigma^{-i+1}(z) \in \mathfrak{m}$. Observe that $\sigma(z), \sigma^{-(j-i)+1}(z) \in \sigma^i(\mathfrak{m})$. This implies that $\sigma^i(\mathfrak{m}) \in \mathscr{M}'_{\mathrm{II}}$, so $\mathscr{M}'_{\mathrm{II}}$ does not have separated chains.

$4 \Rightarrow 3$: Assume that 3 fails; let $\mathfrak{m}$ be a maximal ideal of R with infinite σ-orbit and with $\sigma^i(z), \sigma^j(z), \sigma^k(z) \in \mathfrak{m}$, where $i < j < k$ are integers. We may assume that $j > i$ is minimal such that $\sigma^j(z) \in \mathfrak{m}$ and that $k > j$ is minimal such that $\sigma^k(z) \in \mathfrak{m}$. Let $\mathfrak{n} := \sigma^{-k+1}(\mathfrak{m})$. Observe that $\mathfrak{n} \in \mathscr{M}'_{\mathrm{II}}$ since $\sigma(z) \in \mathfrak{n}$ and $\sigma^{-(k-j)+1}(z) \in \mathfrak{n}$. Since $\sigma^{-(k-i)+1}(z) \in \mathfrak{n}$ as well, with $k - i \neq k - j$, we see that 4 fails.

$3 \Rightarrow 2$: Suppose that $\mathscr{M}'_{\mathrm{II}}$ does not have separated chains. Then there is some $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$ such that $\sigma^{n(\mathfrak{m})}(\mathfrak{m}) \in \mathscr{M}'_{\mathrm{II}}$. Since $\sigma(z), \sigma^{-n(\mathfrak{m})+1}(z), \sigma^{-(n(\mathfrak{m})+n(\sigma^{n(\mathfrak{m})}(\mathfrak{m})))+1}(z) \in \mathfrak{m}$, 3 fails to hold. ■

2.6 Prime Ideals

We now focus on understanding the homogeneous prime ideals of a GWA. Actually, we already have some at hand– the annihilators of the simple modules $V_{\mathfrak{m}}$ defined in section 2.5.1. Our approach will be to seek out strict conditions that force a GWA to have *only* these as its homogeneous prime ideals, with the plan to later address other GWAs by using quotients and localizations to get the strict conditions to hold. The approach taken here was inspired by the work of [34, Section 3].

We will have to take a detour from GWAs to a somewhat more general class of algebras. One problem with GWAs is that a GWA mod a homogeneous ideal is often not a GWA. This makes it difficult to recursively apply our arguments to quotients. Thus we focus on the class of algebras satisfying the following hypotheses:

Hypothesis 46: Let A be a $\mathbb{Z}$-graded k-algebra and let D be a commutative k-algebra. Assume that D is essentially of finite type over k (meaning that it is some localization of an affine commutative k-algebra) and that D is a Jacobson ring (meaning that every prime ideal of D equals the intersection of the maximal ideals containing it). Let $\phi : D \to A_0$ be a surjection of k-algebras, making A a D-D-bimodule. Let σ be an automorphism of D. Assume that whenever $a \in A_m$ and $f \in D$, we have $a.f = \sigma^m(f).a$. Assume that there are $v_m \in A_m$ for all $m \in \mathbb{Z}$ such that $A_m = D.v_m$ (that is, assume that each A_m is cyclic as a left D-module).

These assumptions are almost enough to force A to be a GWA, but the crucial deviation is that σ need not preserve the kernel of ϕ and so σ cannot necessarily be thought of as an automorphism of A_0. The advantage of using Hypothesis 46 is that taking a quotient of A by a homogeneous ideal yields another set of data that satisfies Hypothesis 46.

Assume Hypothesis 46. In this more general context, we shall repeat some of the definitions made in section 2.5.1. There is no ambiguity because the definitions will be equivalent when A is a GWA.

Define for $\mathfrak{m} \in \max\operatorname{spec} D$ the left A-module $M_{\mathfrak{m}} := A/A\mathfrak{m}$. Recall that the notation $\hat{V}(I)$ stands for the set of maximal ideals containing an ideal I. Let $\mathscr{M} = \{\mathfrak{m} \in \hat{V}(\ker(\phi)) \mid \mathfrak{m} \text{ has infinite } \sigma\text{-orbit}\}$.

Proposition 47: *$M_{\mathfrak{m}} \neq 0$ if and only if $\mathfrak{m} \in \hat{V}(\ker(\phi))$*

Proof: We have $M_{\mathfrak{m}} = 0 \Leftrightarrow A\mathfrak{m} = A \Leftrightarrow 1 \in A\mathfrak{m} \Leftrightarrow 1 \in A_0\mathfrak{m}$, and $A_0\mathfrak{m} = \phi(D)\phi(\mathfrak{m}) = \phi(D\mathfrak{m}) = \phi(\mathfrak{m})$, so $M_{\mathfrak{m}} = 0 \Leftrightarrow 1 \in \phi(\mathfrak{m})$. Now if $\phi(f) = 1$ for some $f \in \mathfrak{m}$, then $f - 1 \in \ker(\phi) \setminus \mathfrak{m}$, so $\mathfrak{m} \notin \hat{V}(\ker(\phi))$. Conversely, if $\mathfrak{m} \notin \hat{V}(\ker(\phi))$, then $1 \in A_0 = \phi(D) = \phi(\mathfrak{m} + \ker(\phi)) = \phi(\mathfrak{m})$. So $M_{\mathfrak{m}} = 0 \Leftrightarrow 1 \in \phi(\mathfrak{m}) \Leftrightarrow \mathfrak{m} \notin \hat{V}(\ker(\phi))$. ■

Define $\mathcal{O}$ to be the full subcategory of A-Mod consisting of modules ${}_AM$ such that for some $\mathfrak{m} \in \mathscr{M}$, the D-module ${}_DM$ is a homomorphic image of $\bigoplus_{m\in\mathbb{Z}} D/\sigma^m(\mathfrak{m})$.

Proposition 48: *For $\mathfrak{m} \in \mathscr{M}$, the A-module $M_{\mathfrak{m}}$ is an object of $\mathcal{O}$.*

Proof: We have

$$\begin{aligned} {}_DM_{\mathfrak{m}} &= (\bigoplus_{m\in\mathbb{Z}} A_m)/(\bigoplus_{m\in\mathbb{Z}} A_m\mathfrak{m}) \cong \bigoplus_{m\in\mathbb{Z}} A_m/(A_m\mathfrak{m}) = \bigoplus_{m\in\mathbb{Z}} A_m/(\sigma^m(\mathfrak{m})A_m) \\ &= \bigoplus_{m\in\mathbb{Z}} Dv_m/(\sigma^m(\mathfrak{m})(Dv_m)) = \bigoplus_{m\in\mathbb{Z}} Dv_m/(\sigma^m(\mathfrak{m})v_m). \end{aligned}$$

For any $m \in \mathbb{Z}$, the left D-homomorphism

$$D \xrightarrow{\cdot v_m} Dv_m \twoheadrightarrow Dv_m/(\sigma^m(\mathfrak{m})v_m)$$

descends to a surjection $D/\sigma^m(\mathfrak{m}) \twoheadrightarrow Dv_m/(\sigma^m(\mathfrak{m})v_m)$. It follows that ${}_DM_{\mathfrak{m}}$ is a homomorphic image of $\bigoplus_{m\in\mathbb{Z}} D/\sigma^m(\mathfrak{m})$. ■

Note that $\bigoplus_{m\in\mathbb{Z}} D/\sigma^m(\mathfrak{m})$ is a semisimple D-module, so every submodule of it is a direct summand. It follows that every submodule of $\bigoplus_{m\in\mathbb{Z}} D/\sigma^m(\mathfrak{m})$ is also a homomorphic image of $\bigoplus_{m\in\mathbb{Z}} D/\sigma^m(\mathfrak{m})$, and hence is in $\mathcal{O}$. One easily obtains:

Proposition 49: *$\mathcal{O}$ is closed under subquotients.*

Proposition 50: *For $\mathfrak{m} \in \mathscr{M}$, all submodules of $M_{\mathfrak{m}}$ are homogeneous.*

Proof: Let S be an A-submodule of $M_{\mathfrak{m}}$ and consider any element $\sum_{i\in\mathbb{Z}} a_i$ of S, where $a_i \in (M_{\mathfrak{m}})_i$ for $i \in \mathbb{Z}$. Let $\mathscr{S} = \{i \in \mathbb{Z} \mid a_i \neq 0\}$, a finite set. Then for any $m \in \mathbb{Z}$, there is some $f \in \left(\prod_{k\in\mathscr{S}\setminus\{m\}} \sigma^k(\mathfrak{m})\right)\setminus \sigma^m(\mathfrak{m})$, since $\mathfrak{m}$ has infinite σ-orbit. Now for $k \in \mathscr{S}\setminus\{m\}$ we have $f(M_{\mathfrak{m}})_k = 0$ since $f.A_k \subseteq \sigma^k(\mathfrak{m}).A_k = A_k\mathfrak{m}$. We have $S \ni f\left(\sum_{i\in\mathbb{Z}} a_i\right) = f.a_m$. Since f is a unit mod $\sigma^m(\mathfrak{m})$ and a_m is annihilated by $\sigma^m(\mathfrak{m})$, we have $a_m \in S$. ■

Proposition 51: *For $\mathfrak{m} \in \mathscr{M}$, the A-module $M_{\mathfrak{m}}$ has a unique maximal proper submodule.*

Proof: Let $\mathscr{S}$ be the set of proper submodules of $M_{\mathfrak{m}}$. A submodule of $M_{\mathfrak{m}}$ is proper if and only if it has trivial intersection with $(M_{\mathfrak{m}})_0 = A_0/(A_0\mathfrak{m}) = A_0/\phi(\mathfrak{m}) \cong D/\mathfrak{m}$, for $D/\mathfrak{m}$ is a simple D-module and $M_{\mathfrak{m}}$ is generated by $(M_{\mathfrak{m}})_0$. So $(\sum_{N\in\mathscr{S}} N)_0 = \sum_{N\in\mathscr{S}} N_0 = 0$ (here we are also using Proposition 50). Thus $\sum_{N\in\mathscr{S}} N$ is a proper submodule of $M_{\mathfrak{m}}$. ■

For $\mathfrak{m} \in \mathscr{M}$, we define $V_{\mathfrak{m}}$ to be the unique simple quotient of $M_{\mathfrak{m}}$, and we define $J(\mathfrak{m})$ to be $\operatorname{ann}_A(V_{\mathfrak{m}})$.

Proposition 52: *Let $\mathfrak{m}$ be a maximal ideal of D with infinite σ-orbit, and let ${}_AX$ be a simple A-module with $\operatorname{ann}_X \mathfrak{m}$ being nonzero. Then $\mathfrak{m} \in \mathscr{M}$ and X is isomorphic to $V_{\mathfrak{m}}$.*

Proof: Let $x \in \operatorname{ann}_X \mathfrak{m}$ be nonzero. Then $A \xrightarrow{\cdot x} X$ (the left A-map that sends 1 to x) is surjective (since X is simple) and it descends to a surjective left A-map $M_{\mathfrak{m}} \to X$ (since $A\mathfrak{m}$ is in its kernel). This realizes X as a simple quotient of $M_{\mathfrak{m}}$. Since $M_{\mathfrak{m}}$ could not be zero, we have $\mathfrak{m} \in \mathscr{M}$ by Proposition 47. We are done since $V_{\mathfrak{m}}$ is the unique simple quotient of $M_{\mathfrak{m}}$. ■

Corollary 53: *Any simple object of $\mathcal{O}$ is isomorphic to $V_{\mathfrak{m}}$ for some $\mathfrak{m} \in \mathscr{M}$.*

Proof: Let ${}_AX$ be a simple object of $\mathcal{O}$, say with a left D-linear surjection

$$\psi : \bigoplus_{m\in\mathbb{Z}} D/\sigma^m(\mathfrak{m}) \twoheadrightarrow X,$$

where $\mathfrak{m} \in \mathscr{M}$. Then there is some $m \in \mathbb{Z}$ and some $f \in D/\sigma^m(\mathfrak{m})$ so that $\psi(f) \neq 0$. Then $\psi(f) \in \operatorname{ann}_X(\sigma^m(\mathfrak{m}))$ and $\sigma^m(\mathfrak{m})$ has infinite σ-orbit, so Proposition 52 applies and we get that $X \cong V_{\sigma^m(\mathfrak{m})}$ and that $\sigma^m(\mathfrak{m}) \in \mathscr{M}$. ■

We define the *support* of an A-module M by $\operatorname{Supp}(M) = \{\mathfrak{m} \in \max\operatorname{spec}(D) \mid \operatorname{ann}_M \mathfrak{m} \neq 0\}$. For $\mathfrak{m} \in \mathscr{M}$, the support of $V_{\mathfrak{m}}$ would be $\{\sigma^j(\mathfrak{m}) \mid j \in \mathbb{Z} \text{ and } (V_{\mathfrak{m}})_j \neq 0\}$.

Proposition 54: *For $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathscr{M}$, we have $V_{\mathfrak{m}_1} \cong V_{\mathfrak{m}_2} \Leftrightarrow \operatorname{Supp}(V_{\mathfrak{m}_1}) \cap \operatorname{Supp}(V_{\mathfrak{m}_2}) \neq \emptyset$.*

Proof: The implication $\Rightarrow$ is obvious. Assume for the converse that $\mathfrak{n} \in \operatorname{Supp}(V_{\mathfrak{m}_1}) \cap \operatorname{Supp}(V_{\mathfrak{m}_2})$. Note that $\mathfrak{n}$ has infinite σ-orbit, since it is $\sigma^j(\mathfrak{m}_1)$ for some $j \in \mathbb{Z}$. Since $\operatorname{ann}_{V_{\mathfrak{m}_i}}(\mathfrak{n})$ is nonzero for $i \in \{1, 2\}$, Proposition 52 implies that $V_{\mathfrak{m}_1} \cong V_{\mathfrak{n}} \cong V_{\mathfrak{m}_2}$. ■

Before proving the main theorem of this section, we will need a technical lemma.

2.6.1 Preparation for the Main Theorem

The main lemma proven in this section is essentially [34, Lemma 3.2.1]. However, we provide a detailed treatment here both for completeness and for the sake of allowing the lemma to apply to the more general setting that we need.

For the proof we will need the notion of GK-homogeneity from [27, Chapter 5]: a module over a k-algebra is said to be *GK-homogeneous* when all its nonzero submodules have the same GK-dimension.

Proposition 55: *If R is a commutative noetherian k-algebra such that ${}_RR$ is GK-homogeneous and has finite GK-dimension, then a prime ideal P of R is a minimal prime if and only if* $\mathrm{GK}(R/P) = \mathrm{GK}(R)$.

Proof: If P is a non-minimal prime of R then there is some prime $Q \subsetneq P$. Then $\mathrm{GK}(R/P) = \mathrm{GK}((R/Q)/(P/Q)) \leq \mathrm{GK}(R/Q) - 1$ by [32, 8.3.6(i)]. We also have $\mathrm{GK}(R/Q) \leq \mathrm{GK}(R)$ by [32, 8.3.2(ii)]. Thus $\mathrm{GK}(R/P) < \mathrm{GK}(R)$.

Suppose for the converse that P is a minimal prime of R. By [26, Theorem 86], there is some nonzero $r \in R$ such that $P = \mathrm{ann}_R(r)$. Since ${}_RR/P \cong Rr$ and ${}_RR$ is GK-homogeneous, we conclude that $\mathrm{GK}(R/P) = \mathrm{GK}(R)$. ■

Proposition 56: *Assume that R is commutative and essentially of finite type over k. Then* $\mathrm{GK}(R) = \mathrm{GK}(R/P)$ *for some minimal prime P of R.*

Proof: Let R be $T\mathcal{E}^{-1}$, where T is a commutative affine k-algebra and $\mathcal{E}$ is a multiplicative subset of T. Without loss of generality, we may take $\mathcal{E}$ to consist of only regular elements (replace T by a suitable quotient). Using [27, Theorem 4.5], we have that $\mathrm{GK}(T) = \mathrm{GK}(T/P)$ for some minimal prime P of T. Since $\mathcal{E}$ consists of only regular elements, it is disjoint from P, and so $P\mathcal{E}^{-1}$ is clearly a minimal prime of R. Finally, using [27, Proposition 4.2], we have

$$\mathrm{GK}(R) = \mathrm{GK}(T\mathcal{E}^{-1}) = \mathrm{GK}(T) = \mathrm{GK}(T/P) = \mathrm{GK}(T\mathcal{E}^{-1}/P\mathcal{E}^{-1}) = \mathrm{GK}(R/(P\mathcal{E}^{-1})).$$ ■

Lemma 57: *Assume that R is a commutative noetherian k-algebra and that S is commutative and essentially of finite type over k. Assume also that R and S are GK-homogeneous as modules over themselves and that they have the same GK dimension. Let $\phi : R \to S$ be a surjective algebra homomorphism. Then* $\phi(\mathrm{rad}(R)) = \mathrm{rad}(S)$*, where* $\mathrm{rad}(\cdot)$ *denotes the nilradical of a ring.*

Proof: The inclusion $\phi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$ is automatic: the preimage under ϕ of a semiprime ideal is semiprime, so in particular $\phi^{-1}(\mathrm{rad}(S))$ is semiprime. It follows that $\phi^{-1}(\mathrm{rad}(S)) \supseteq \mathrm{rad}(R)$, and hence that $\phi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$. Now by passing to the quotient by $\mathrm{rad}(R)$ we may assume that R is semiprime. Our goal is now to show that S is semiprime.

We first show that minimal primes pull back to minimal primes under ϕ. Let P be a minimal prime ideal of R. Note that Proposition 55 applies to both R and S. We have $R/\phi^{-1}(P) \cong S/P$, so $\mathrm{GK}(R/\phi^{-1}(P)) = \mathrm{GK}(S/P) = \mathrm{GK}(S) = \mathrm{GK}(R)$, and it follows that $\phi^{-1}(P)$ is a minimal prime of R.

We now show that ϕ takes regular elements to regular elements. Suppose that $t \in R$ and $\phi(t)$ is not regular. Choose a nonzero $s \in S$ such that $s\phi(t) = 0$. Let $I = \mathrm{ann}_S(s)$. Since S is GK-homogeneous and $S/I \cong Ss$ as S-modules, we have $\mathrm{GK}(S) = \mathrm{GK}(S/I)$. Applying Proposition 56 to S/I, we have $\mathrm{GK}(S/I) = \mathrm{GK}(S/P)$ for some prime P minimal over I. Since $\mathrm{GK}(S) = \mathrm{GK}(S/P)$, Proposition 55 tells us that P is a minimal prime of S. Thus $\phi(t) \in P$ for a minimal prime P of S. The argument above showed that $\phi^{-1}(P)$ is a minimal prime of R, so $t \in \phi^{-1}(P)$ cannot be regular. Thus ϕ takes regular elements to regular elements.

Let R' and S' respectively be the localizations of R and S at their sets of regular elements. Then R' is semiprime and artinian, so it is a product of fields. Since ϕ induces a surjective homomorphism $R' \to S'$, it follows that S' is also a product of fields and is therefore semiprime. Since $S \subseteq S'$, we conclude that S is semiprime. ■

The following appears as [34, Lemma 3.2.1] with slightly different hypotheses.

Lemma 58: *Let $A = \bigoplus_{m\in\mathbb{Z}}$ be a $\mathbb{Z}$-graded k-algebra. Assume that A is graded-prime and graded-left noetherian. Assume that for each $m \in \mathbb{Z}$, there is some $v_m \in A_m$ such that $A_m = v_m A_0 = A_0 v_m$. Assume that A_0 is commutative and essentially of finite type over k. Then A_0 is semiprime, and for all $m \in \mathbb{Z}$ the A_0-module A_m, viewed either as a left or as a right A_0-module, is a semiprime quotient of A_0.*

Proof: We first show that all the A_m are GK-homogeneous and have the same GK-dimension. Let $t = \mathrm{GK}(A_0)$. For $m \in \mathbb{Z}$, let I_m denote the unique left A_0-submodule of A_m that is maximal among submodules that have GK-dimension less than t.

> **Claim:** For $m \in \mathbb{Z}$, the submodule I_m is also maximal as a *right* A_0-submodule of A_m that has GK-dimension less than t.
> **Proof:** For any $x \in A_0$, the left A_0-submodule $I_m x$ of A_m, being a homomorphic image of I_m, has GK-dimension less than t. It follows that $I_m x \subseteq I_m$, so I_m is a right A_0-submodule of A_m. Let J_m denote the unique maximal right A_0-submodule of A_m that has GK-dimension less than t. Note that ${}_{A_0}(I_m)$ and $(I_m)_{A_0}$ are both homomorphic images of ideals of A_0 and are therefore finitely generated modules. Since I_m is an A_0-A_0 bimodule that is finitely generated on both sides, it follows from [27, Corollary 5.4] that the GK-dimension of I_m as a right A_0-module equals its GK-dimension as a left A_0-module. Since I_m is a right A_0 submodule of A_m and has GK-dimension less than t, we have $I_m \subseteq J_m$. Applying the same arguments with a left-right reversal leads to $J_m \subseteq I_m$.

For any $m_1, m_2 \in \mathbb{Z}$ we have that $I_{m_1} A_{m_2} = I_{m_1} v_{m_2}$ is a homomorphic image of ${}_{A_0}(I_{m_1})$ and hence has GK-dimension less than t. It follows that $I_{m_1} A_{m_2} \subseteq I_{m_1+m_2}$, and similarly we get $A_{m_2} I_{m_1} \subseteq I_{m_1+m_2}$. Hence $I := \bigoplus_{m\in\mathbb{Z}} I_m$ is a homogeneous ideal of A. Let $J = \mathrm{r.ann}_A(I)$. Since A is graded-left noetherian, we have $I = Ax_1 + \cdots + Ax_n$ for some homogeneous left generators x_i. Choose m_i so that $x_i \in I_{m_i}$ for $i \in \{1, \ldots, n\}$. Since $A_0/\mathrm{r.ann}_{A_0}(x_i)$ embeds into I_{m_i} as a right A_0-module, it has GK-dimension less than t. By [32, Proposition 8.3.2(iv)], it follows that $J_0 = \bigcap_i \mathrm{r.ann}_{A_0}(x_i)$ is nonzero. Since A is graded-prime and $IJ = 0$, we get $I = 0$. This shows that every nonzero left or right A_0-submodule of each A_m has GK-dimension at least t. Since each A_m is a homomorphic image of A_0 as a left and as a right A_0-module, we obtain equality: all the A_m are GK-homogeneous as left or right A_0-modules, and they all have GK-dimension t.

For $m \in \mathbb{Z}$, let $X_m = \mathrm{l.ann}_{A_0}(v_m)$ and let $Y_m = \mathrm{r.ann}_{A_0}(v_m)$. Let $B_m = A_0/X_m$ and let $C_m = A_0/Y_m$. Define a family of isomorphisms $\sigma_m : C_m \to B_m$ by $v_m c = \sigma_m(c) v_m$. Let $\delta : A_0 \to B_m$ and $\epsilon : A_0 \to C_m$ be the quotient maps. Note that ${}_{A_0}B_m \cong {}_{A_0}A_m$ and $(C_m)_{A_0} \cong (A_m)_{A_0}$, so Lemma 57 applies to δ and ϵ. Using Lemma 57, we have

$$A_m(\mathrm{rad}A_0) = A_m\epsilon(\mathrm{rad}A_0) = A_m(\mathrm{rad}C_m) = \sigma_m(\mathrm{rad}C_m)A_m = (\mathrm{rad}B_m)A_m = \delta(\mathrm{rad}A_0)A_m = (\mathrm{rad}A_0)A_m$$

for $m \in \mathbb{Z}$. Thus $A\mathrm{rad}(A_0) = \mathrm{rad}(A_0)A$ is an ideal of A. It is easy to see that this is a nilpotent ideal, so since A is graded-prime we conclude that $\mathrm{rad}(A_0) = 0$. By Proposition 57 we also conclude that $\mathrm{rad}(B_m)$ and $\mathrm{rad}(C_m)$ are 0 for all $m \in \mathbb{Z}$. ■

2.6.2 The Main Theorem

We are finally equipped to prove our main theorem on prime ideals in GWAs, Theorem 60 and its later specialization to GWAs, Theorem 70.

We take the *length* of a module to be its composition length, when it is finite, and we take the length of a module to be ∞ otherwise.

Lemma 59: *Assume Hypothesis 46, and assume the following:*

1. *A is graded left noetherian.*
2. *There is a uniform finite upper bound on the length of $M_\mathfrak{m}$ for $\mathfrak{m} \in \mathscr{M}$.*
3. *The set $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\}$ is finite.*
4. *$\mathscr{M} = \hat{V}(\ker(\phi))$. That is, all $\mathfrak{m} \in \hat{V}(\ker(\phi))$ have infinite σ-orbit.*
5. *A is graded-prime.*

Then $0 = J(\mathfrak{m})$ for some $\mathfrak{m} \in \mathscr{M}$.

Proof: For any $\mathfrak{m} \in \mathscr{M}$, we have $\mathrm{ann}_A(M_\mathfrak{m}) \subseteq A\mathfrak{m}$. Let us show that these intersect to zero.

> **Claim:** $\bigcap_{\mathfrak{m}\in\mathscr{M}} A\mathfrak{m} = 0$.
> **Proof:** Since $\bigcap_{\mathfrak{m}\in\mathscr{M}} A\mathfrak{m}$ is a homogeneous subspace, we can show that it is zero by showing that it is zero in each degree. Its component in a given degree $m \in \mathbb{Z}$ is $\bigcap_{\mathfrak{m}\in\mathscr{M}} A_m\mathfrak{m}$. Note that Hypothesis 46 does imply that $A_m = A_0 v_m = v_m A_0$ for $m \in \mathbb{Z}$, so Lemma 58 applies and we have that A_0 is semiprime and that each A_m is isomorphic to a semiprime quotient of A_0 as a left and as a right A_0-module. Let $\psi' : A_0 \twoheadrightarrow A_m$ be a right A_0-homomorphism with kernel a semiprime ideal of A_0. Let $\psi : D \twoheadrightarrow A_m$ be $\psi' \circ \phi$; then this is a right D-homomorphism with kernel a semiprime ideal of D. For $\mathfrak{m} \in \mathscr{M}$, we have $\psi^{-1}(A_m\mathfrak{m}) = \mathfrak{m}$ if $\ker(\psi) \subseteq \mathfrak{m}$ and $\psi^{-1}(A_m\mathfrak{m}) = D$ otherwise. Thus
> $$\psi^{-1}\left(\bigcap_{\mathfrak{m}\in\mathscr{M}} A_m\mathfrak{m}\right) = \bigcap_{\mathfrak{m}\in\mathscr{M}} \psi^{-1}(A_m\mathfrak{m}) = \bigcap_{\mathfrak{m}\in\mathscr{M}\cap\hat{V}(\ker(\psi))} \mathfrak{m}.$$
> Now $\mathscr{M}\cap\hat{V}(\ker(\psi)) = \hat{V}(\ker(\psi))$, since $\hat{V}(\ker(\psi)) \subseteq \hat{V}(\ker(\phi)) = \mathscr{M}$. So since $\ker(\psi)$ is semiprime and A_0 is a Jacobson ring, we get:
> $$\psi^{-1}\left(\bigcap_{\mathfrak{m}\in\mathscr{M}} A_m\mathfrak{m}\right) = \bigcap_{\mathfrak{m}\in\hat{V}(\ker(\psi))} \mathfrak{m} = \ker(\psi);$$
> in other words we get that $\bigcap_{\mathfrak{m}\in\mathscr{M}} A_m\mathfrak{m} = 0$.

Hence $\bigcap_{\mathfrak{m}\in\mathscr{M}} \mathrm{ann}_A(M_\mathfrak{m}) = 0$. Let N be a uniform finite upper bound on the length of the $M_\mathfrak{m}$, for $\mathfrak{m} \in \mathscr{M}$. For each $\mathfrak{m} \in \mathscr{M}$, the left A-module $M_\mathfrak{m}$ has a composition series of length at most N. Consider the composition factors. They are subquotients of $M_\mathfrak{m}$, and hence they are simple objects of $\mathcal{O}$ (since $M_\mathfrak{m}$ is an object of $\mathcal{O}$ and $\mathcal{O}$ is closed under subquotients). By Corollary 53, each composition factor is isomorphic to a $V_\mathfrak{n}$ for some $\mathfrak{n} \in \mathscr{M}$. The product of their annihilators should annihilate $M_\mathfrak{m}$. Thus, for each $\mathfrak{m} \in \mathscr{M}$, there are $\mathfrak{n}_1^\mathfrak{m}, \dots, \mathfrak{n}_{k_\mathfrak{m}}^\mathfrak{m} \in \mathscr{M}$ such that

$$J(\mathfrak{n}_1^\mathfrak{m}) \cdots J(\mathfrak{n}_{k_\mathfrak{m}}^\mathfrak{m}) \subseteq \mathrm{ann}_A(M_\mathfrak{m}),$$

where $k_{\mathfrak{m}} \leq N$. Since $\bigcap_{\mathfrak{m}\in\mathscr{M}} \operatorname{ann}_A(M_{\mathfrak{m}}) = 0$, it follows that

$$0 = \bigcap_{\mathfrak{m}\in\mathscr{M}} J(\mathfrak{n}_1^{\mathfrak{m}}) \cdots J(\mathfrak{n}_{k_{\mathfrak{m}}}^{\mathfrak{m}}).$$

Each $k_{\mathfrak{m}}$ is at most N, and there are only finitely many distinct $J(\mathfrak{n})$. It follows that there are only finitely many possible products $J(\mathfrak{n}_1^{\mathfrak{m}}) \cdots J(\mathfrak{n}_{k_{\mathfrak{m}}}^{\mathfrak{m}})$. This means that the intersection above can be done over only finitely many $\mathfrak{m} \in \mathscr{M}$. Since the $J(\mathfrak{n})$ are homogeneous ideals and A is graded-prime, it follows that some $J(\mathfrak{n})$ is zero. ∎

At this point, we will take advantage of our detour from GWAs and benefit from the fact that Lemma 59 can be applied recursively to quotients. The following theorem may be compared to [34, Theorem 3.2.4].

Theorem 60: *Assume Hypothesis 46 and assume the following:*

1. *A is graded left noetherian.*
2. *There is a uniform finite upper bound on the length of $M_{\mathfrak{m}}$ for $\mathfrak{m} \in \mathscr{M}$.*
3. *The set $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\}$ is finite.*
4. *$\mathscr{M} = \hat{V}(\ker(\phi))$. That is, all $\mathfrak{m} \in \hat{V}(\ker(\phi))$ have infinite σ-orbit.*

Then every homogeneous prime ideal of A is $J(\mathfrak{m})$ for some $\mathfrak{m} \in \mathscr{M}$.

Proof: Let P be any homogeneous prime ideal of A; define $A' = A/P$. We will prove the theorem by showing that the graded-prime algebra $A' = A/P$ satisfies the hypotheses of Lemma 59. First, A' is $\mathbb{Z}$-graded, and we identify it with $\bigoplus_{m\in\mathbb{Z}} A_m/P_m$. Let ϕ' be the k-algebra surjection

$$D \xrightarrow{\phi} A_0 \xrightarrow{\text{quo}} A_0/P_0.$$

It is easy to see that the objects A', D, ϕ', and σ satisfy Hypothesis 46. We also have that A' is graded left noetherian. Let $\mathscr{M}' = \{\mathfrak{m} \in \hat{V}(\ker(\phi')) \mid \mathfrak{m} \text{ has infinite } \sigma\text{-orbit}\}$ and let $M'_{\mathfrak{m}}$ be the left A'-module $A'/A'\mathfrak{m}$, for any $\mathfrak{m} \in \mathscr{M}'$. Since $\ker(\phi) \subseteq \ker(\phi')$ and $\mathscr{M} = \hat{V}(\ker(\phi))$, we get that $\mathscr{M}' = \hat{V}(\ker(\phi'))$. For any $\mathfrak{m} \in \mathscr{M}'$, the module $M'_{\mathfrak{m}} \cong (A/A\mathfrak{m})/(P/A\mathfrak{m})$ is a quotient of $M_{\mathfrak{m}}$, so its unique simple quotient is isomorphic to $V_{\mathfrak{m}}$, the unique simple quotient of $M_{\mathfrak{m}}$. Thus the annihilator in A' of the unique simple quotient of each $M'_{\mathfrak{m}}$, for $\mathfrak{m} \in \mathscr{M}' \subseteq \mathscr{M}$, is $J(\mathfrak{m})/P$; so there are finitely many distinct ones. Since the $M'_{\mathfrak{m}}$ are quotients of the $M_{\mathfrak{m}}$, the length of the $M'_{\mathfrak{m}}$ is uniformly bounded over $\mathfrak{m} \in \mathscr{M}' \subseteq \mathscr{M}$. Lemma 59 now implies that $J(\mathfrak{m})/P = 0$ for some $\mathfrak{m} \in \mathscr{M}' \subseteq \mathscr{M}$. ∎

The hypotheses of this theorem are not very friendly to check. Let us specialize back to GWAs and find more manageable conditions that are equivalent to these hypotheses.

Hypothesis 61: Let $A = D[x, y; \sigma, z]$ be a GWA. Assume that D is a commutative k-algebra that is essentially of finite type over k and that is a Jacobson ring.

Obviously,

Proposition 62: *Hypothesis 61 implies Hypothesis 46, with ϕ taken to be $id: D \to D$.*

In the Hypothesis 61 setting, we adopt the same notations developed above for the Hypothesis 46 setting ($\mathscr{M}$, $M_{\mathfrak{m}}$, $V_{\mathfrak{m}}$, $J(\mathfrak{m})$, etc.). We will also use the notation of Definitions 29 and 33. Note that the use of $\mathscr{M}$, $M_{\mathfrak{m}}$, and $V_{\mathfrak{m}}$ in section 2.5.1 agrees with their current usage in the Hypothesis 61 setting. For the sake of having consistent notation throughout this section, an integer subscript on a homogeneous ideal of a GWA will continue to denote the corresponding graded component (rather than following the convention of Definition 13).

Proposition 63: *Assume Hypothesis 61. The following are equivalent:*

1. *There is a uniform finite upper bound on the length of the $M_{\mathfrak{m}}$, for $\mathfrak{m} \in \mathscr{M}$.*
2. *There is a uniform finite upper bound on the cardinality of the sets $\{i \in \mathbb{Z} \mid \sigma^i(z) \in \mathfrak{m}\}$, for $\mathfrak{m} \in \mathscr{M}$.*

Proof: Fix $\mathfrak{m} \in \mathscr{M}$. In this setting we know all the submodules of $M_{\mathfrak{m}}$ due to Lemma 32. If $\{i \in \mathbb{Z} \mid \sigma^i(z) \in \mathfrak{m}\}$ is finite with cardinality N, then $M_{\mathfrak{m}}$ has at most $2 + N + N + N^2$ submodules (adding up the possibilities for the four different types described in Lemma 32). Therefore 2⇒1.

For the converse, assume that 2 fails. Let $N \in \mathbb{Z}_{\geq 0}$; we will see that some $M_{\mathfrak{m}}$ has length exceeding N. Either there is some $\mathfrak{m} \in \mathscr{M}$ for which $\{i \leq 0 \mid \sigma^i(z) \in \mathfrak{m}\}$ has at least N elements, or there is some $\mathfrak{m} \in \mathscr{M}$ for which $\{i > 0 \mid \sigma^i(z) \in \mathfrak{m}\}$ has at least N elements. In the first case one can build a properly descending chain of submodules of type 2 (from the enumeration of Lemma 32) which exceeds N in length. In the second case one can build a properly descending chain of submodules of type 3 which exceeds N in length. ∎

Running over all $\mathfrak{m} \in \mathscr{M}$, many of the $J(\mathfrak{m})$ are actually equal. The following proposition will help us sift through them.

Proposition 64: *In the Hypothesis 61 setting, if $\mathfrak{m} \in \mathscr{M}_{II}$, then $J(\mathfrak{m}) = J(\mathfrak{n})$ for some $\mathfrak{n} \in \mathscr{M}'_{II}$.*

Proof: By Proposition 36, the support of $V_{\mathfrak{m}}$ contains $\sigma^{1-n'(\mathfrak{m})}(\mathfrak{m})$. Proposition 54 now tells us that $J(\mathfrak{m}) = J(\sigma^{1-n'(\mathfrak{m})}(\mathfrak{m}))$. It is clear that $\mathfrak{n} = \sigma^{1-n'(\mathfrak{m})}(\mathfrak{m})$ has the desired properties. ∎

The following proposition further helps to describe the $J(\mathfrak{m})$, and it applies in the more general setting (c.f. [34, Proposition 3.2.2]). For a set of maximal ideals $S \subset \operatorname{max\,spec} D$ we denote by $I(S)$ the intersection $\bigcap S$ (the ideal of "functions" vanishing at all "points" in S). We denote by $\overline{S}$ the closure of S in terms of the Zariski topology on $\operatorname{max\,spec} D$.

Proposition 65: *Assume Hypothesis 46. Then*

1. *For any $\mathfrak{m} \in \mathscr{M}$, we have*
$$J(\mathfrak{m}) = \bigoplus_{m \in \mathbb{Z}} I(\sigma^m(\operatorname{Supp} V_{\mathfrak{m}}) \cap \operatorname{Supp}(V_{\mathfrak{m}}))v_m,$$
where $\sigma^m(\operatorname{Supp} V_{\mathfrak{m}})$ denotes $\{\sigma^m(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Supp} V_{\mathfrak{m}}\}$.
2. *If $\mathfrak{m} \in \mathscr{M}$ such that*
$$I(\overline{\operatorname{Supp} V_{\mathfrak{m}} \cap \sigma^m(\operatorname{Supp} V_{\mathfrak{m}})}) = I(\overline{\operatorname{Supp} V_{\mathfrak{m}}}) + I(\overline{\sigma^m(\operatorname{Supp} V_{\mathfrak{m}})}), \tag{28}$$
for all $m \in \mathbb{Z}$, then $J(\mathfrak{m})_0$ is semiprime and $\hat{V}(\phi^{-1}(J(\mathfrak{m})_0)) = \overline{\operatorname{Supp} V_{\mathfrak{m}}}$. And for $m \in \mathbb{Z}$ we have $J(\mathfrak{m})_m = J(\mathfrak{m})_0 v_m + v_m J(\mathfrak{m})_0$.
3. *If $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathscr{M}$ and $\mathfrak{m}_1, \mathfrak{m}_2$ both satisfy (28) for all $m \in \mathbb{Z}$, then $J(\mathfrak{m}_1) = J(\mathfrak{m}_2) \Leftrightarrow \overline{\operatorname{Supp} V_{\mathfrak{m}_1}} = \overline{\operatorname{Supp} V_{\mathfrak{m}_2}}$.*

Proof: Fix $\mathfrak{m} \in \mathscr{M}$. The ideal $J(\mathfrak{m})$ is homogeneous, so it can be written as $\bigoplus_{m \in \mathbb{Z}} I_m.v_m$, where each I_m is an ideal of D containing $\ker(\phi)$. Specifically,

$$I_m = \{f \in D \mid (f.v_m)(V_{\mathfrak{m}})_k = 0 \ \forall k \in \mathbb{Z}\}$$

for $m \in \mathbb{Z}$.

Claim: For $m \in \mathbb{Z}$, $v_m(V_\mathfrak{m})_k$ is nonzero if and only if $\sigma^k(\mathfrak{m}), \sigma^{k+m}(\mathfrak{m}) \in \operatorname{Supp} V_\mathfrak{m}$, and in this case we have $v_m(V_\mathfrak{m})_k = (V_\mathfrak{m})_{m+k}$.

Proof: If $\sigma^k(\mathfrak{m}) \notin \operatorname{Supp} V_\mathfrak{m}$ then $(V_\mathfrak{m})_k = 0$, so $v_m(V_\mathfrak{m})_k = 0$. If $\sigma^{k+m}(\mathfrak{m}) \notin \operatorname{Supp} V_\mathfrak{m}$ then $(V_\mathfrak{m})_{k+m} = 0$, so $v_m(V_\mathfrak{m})_k \subseteq (V_\mathfrak{m})_{k+m} = 0$. This shows one direction; for the converse assume that $\sigma^k(\mathfrak{m}), \sigma^{k+m}(\mathfrak{m}) \in \operatorname{Supp} V_\mathfrak{m}$. Then $(V_\mathfrak{m})_k \neq 0$ and $(V_\mathfrak{m})_{k+m} \neq 0$ so

$$\begin{aligned} 0 \neq (V_\mathfrak{m})_{k+m} &= (A(V_\mathfrak{m})_k)_{k+m} = A_m(V_\mathfrak{m})_k \\ &= (A_0 v_m)(V_\mathfrak{m})_k. \end{aligned}$$

It follows that $v_m(V_\mathfrak{m})_k \neq 0$. In this situation we indeed have $(V_\mathfrak{m})_{m+k} = v_m(V_\mathfrak{m})_k$, because the latter is a nonzero D-submodule of the simple left D-module $(V_\mathfrak{m})_{m+k}$.

Now we have

$$\begin{aligned} I_m &= \{f \in D \mid \forall k \in \mathbb{Z} \text{ with } \sigma^k(\mathfrak{m}), \sigma^{k+m}(\mathfrak{m}) \in \operatorname{Supp} V_\mathfrak{m},\ f.(V_\mathfrak{m})_{m+k} = 0\} \\ &= \{f \in D \mid \forall k \in \mathbb{Z} \text{ with } \sigma^k(\mathfrak{m}), \sigma^{k+m}(\mathfrak{m}) \in \operatorname{Supp} V_\mathfrak{m},\ f \in \sigma^{k+m}(\mathfrak{m})\} \\ &= \bigcap \{\sigma^{k+m}(\mathfrak{m}) \mid k \in \mathbb{Z} \text{ and } \sigma^k(\mathfrak{m}), \sigma^{k+m}(\mathfrak{m}) \in \operatorname{Supp} V_\mathfrak{m}\ \} \\ &= \bigcap (\{\sigma^{k+m}(\mathfrak{m}) \mid k \in \mathbb{Z} \text{ and } \sigma^k(\mathfrak{m}) \in \operatorname{Supp} V_\mathfrak{m}\ \} \cap \{\sigma^{k+m}(\mathfrak{m}) \mid k \in \mathbb{Z} \text{ and } \sigma^{k+m}(\mathfrak{m}) \in \operatorname{Supp} V_\mathfrak{m}\ \}) \\ &= \bigcap (\sigma^m(\operatorname{Supp} V_\mathfrak{m}) \cap \operatorname{Supp} V_\mathfrak{m}) \\ &= I\left(\sigma^m(\operatorname{Supp} V_\mathfrak{m}) \cap \operatorname{Supp} V_\mathfrak{m}\right), \end{aligned}$$

proving assertion 1 of the Proposition. Notice that the fourth line uses the fact that $\mathfrak{m}$ has infinite σ-orbit. Continuing the calculation while making the assumption (28), we get

$$\begin{aligned} I_m &= I\left(\overline{\sigma^m(\operatorname{Supp} V_\mathfrak{m}) \cap \operatorname{Supp} V_\mathfrak{m}}\right) \\ &= I\left(\overline{\sigma^m(\operatorname{Supp} V_\mathfrak{m})}\right) + I\left(\overline{\operatorname{Supp} V_\mathfrak{m}}\right) \\ &= \sigma^m(I(\overline{\operatorname{Supp} V_\mathfrak{m}})) + I\left(\overline{\operatorname{Supp} V_\mathfrak{m}}\right). \end{aligned}$$

for $m \in \mathbb{Z}$. The final line uses the fact that $\mathfrak{m} \mapsto \sigma^m(\mathfrak{m})$ is an isomorphism of varieties to pull the σ^m through the closure operation. A special case of the above is that $I_0 = I\left(\overline{\operatorname{Supp} V_\mathfrak{m}}\right)$, so $J(\mathfrak{m})_0 = \phi(I_0)$ is semiprime and $\hat{V}(\phi^{-1}(J(\mathfrak{m})_0)) = \hat{V}(I_0) = \overline{\operatorname{Supp} V_\mathfrak{m}}$. Other $m \in \mathbb{Z}$ can be viewed in terms of the $m = 0$ case:

$$\begin{aligned} I_m &= \sigma^m(I(\overline{\operatorname{Supp} V_\mathfrak{m}})) + I\left(\overline{\operatorname{Supp} V_\mathfrak{m}}\right) \\ &= \sigma^m(I_0) + I_0. \end{aligned}$$

It follows that $J(\mathfrak{m})_m = (\sigma^m(I_0) + I_0).v_m = v_m.I_0 + I_0.v_m = v_m J(\mathfrak{m})_0 + J(\mathfrak{m})_0 v_m$, for $m \in \mathbb{Z}$. Thus we have proven assertion 2 of the Proposition. Notice that this implies that $J(\mathfrak{m})$ is generated as an ideal in degree zero. For the last part of the Proposition, assume that $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathscr{M}$ satisfy the criterion (28) for all $m \in \mathbb{Z}$. Assume that $J(\mathfrak{m}_1) = J(\mathfrak{m}_2)$. Then $\overline{\operatorname{Supp} V_{\mathfrak{m}_i}} = \hat{V}(\phi^{-1}(J(\mathfrak{m}_i)_0))$ for $i \in \{1, 2\}$, so $\overline{\operatorname{Supp} V_{\mathfrak{m}_1}} = \overline{\operatorname{Supp} V_{\mathfrak{m}_2}}$. Conversely, assume that $\overline{\operatorname{Supp} V_{\mathfrak{m}_1}} = \overline{\operatorname{Supp} V_{\mathfrak{m}_2}}$. Then $\phi^{-1}(J(\mathfrak{m}_i)_0) = I(\hat{V}(\phi^{-1}(J(\mathfrak{m}_i)_0))) = I(\overline{\operatorname{Supp} V_{\mathfrak{m}_i}})$, so $J(\mathfrak{m}_1)_0 = J(\mathfrak{m}_2)_0$. Since $J(\mathfrak{m}_1), J(\mathfrak{m}_2)$ are generated in degree zero, this gives $J(\mathfrak{m}_1) = J(\mathfrak{m}_2)$. ∎

We can now give a fairly explicit description of the $J(\mathfrak{m})$ in the GWA setting, and we are about to obtain a much friendlier criterion to replace hypothesis 3 of Theorem 60.

Proposition 66: *Assume Hypothesis 61. For any $\mathfrak{m} \in \mathscr{M}$,*

$$J(\mathfrak{m}) = \bigoplus_{m \in \mathbb{Z}} \left(\bigcap \{\sigma^j(\mathfrak{m}) \mid -n'(\mathfrak{m}) < j < n(\mathfrak{m}) \text{ and } -n'(\mathfrak{m}) + m < j < n(\mathfrak{m}) + m\}\right) v_m.$$

Proof: Use assertion 1 of Proposition 65 and use Proposition 36. ∎

Proposition 67: *Assume Hypothesis 61. For* $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathscr{M}_I$*, we have* $J(\mathfrak{m}_1) = J(\mathfrak{m}_2) \Leftrightarrow \overline{\operatorname{Supp} V_{\mathfrak{m}_1}} = \overline{\operatorname{Supp} V_{\mathfrak{m}_2}}$.

Proof: By assertion 3 of Proposition 65, the present Proposition is established if we can show that (28) is satisfied for all $\mathfrak{m} \in \mathscr{M}_{\mathrm{I}}$ and $m \in \mathbb{Z}$. Fix $\mathfrak{m} \in \mathscr{M}_{\mathrm{I}}$ and $m \in \mathbb{Z}$. Using Proposition 36 and the fact that one of $n(\mathfrak{m}), n'(\mathfrak{m})$ is infinite, one of $\operatorname{Supp} V_{\mathfrak{m}}$ and $\sigma^m(\operatorname{Supp} V_{\mathfrak{m}})$ must be contained in the other. The criterion (28) follows easily. ■

Proposition 68: *Assume Hypothesis 61. For* $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathscr{M}'_{II}$*, we have* $J(\mathfrak{m}_1) = J(\mathfrak{m}_2) \Leftrightarrow \mathfrak{m}_1 = \mathfrak{m}_2$.

Proof: Assume that $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathscr{M}'_{\mathrm{II}}$ and $J(\mathfrak{m}_1) = J(\mathfrak{m}_2)$. By Proposition 66 we have

$$J(\mathfrak{m}_i)_m = \bigcap\{\sigma^j(\mathfrak{m}_i) \mid 0 \leq j < n(\mathfrak{m}_i) \text{ and } m \leq j < n(\mathfrak{m}_i) + m\} v_m \tag{29}$$

for $m \in \mathbb{Z}$ and $i \in \{1, 2\}$. It follows that $n(\mathfrak{m}_i) = \min\{\ell > 0 \mid J(\mathfrak{m}_i)_\ell = D\}$ for $i \in \{1, 2\}$, so $n(\mathfrak{m}_1) = n(\mathfrak{m}_2)$. Let $n := n(\mathfrak{m}_1) = n(\mathfrak{m}_2)$. From (29) we also have that $J(\mathfrak{m}_i)_{n-1} = \sigma^{n-1}(\mathfrak{m}_i) v_{n-1}$ for $i \in \{1, 2\}$. It follows that $\sigma^{n-1}(\mathfrak{m}_1) = \sigma^{n-1}(\mathfrak{m}_2)$, and thus that $\mathfrak{m}_1 = \mathfrak{m}_2$. ■

Proposition 69: *Assume Hypothesis 61. The following are equivalent:*

1. *The set* $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\}$ *is finite.*
2. *The sets* $\mathscr{M}'_{II}$ *and* $\{\overline{\operatorname{Supp} V_{\mathfrak{m}}} \mid \mathfrak{m} \in \mathscr{M}_I\}$ *are finite.*

Proof: Since $\mathscr{M} = \mathscr{M}_{\mathrm{I}} \cup \mathscr{M}_{\mathrm{II}}$, the set $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\}$ is finite if and only if the sets $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\}$ and $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{II}}\}$ are finite. By Proposition 67, the set $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\}$ is finite if and only if $\{\overline{\operatorname{Supp} V_{\mathfrak{m}}} \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\}$ is finite. By Proposition 64, we have $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{II}}\} = \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}\}$. By Proposition 68, the latter set is finite if and only if $\mathscr{M}'_{\mathrm{II}}$ is finite. ■

Let us restate Theorem 60 in terms of GWAs, with the nicer versions of the hypotheses.

Theorem 70: *Assume Hypothesis 61 and assume the following:*

1. *The sets* $\mathscr{M}'_{II}$ *and* $\{\overline{\operatorname{Supp} V_{\mathfrak{m}}} \mid \mathfrak{m} \in \mathscr{M}_I\}$ *are finite.*
2. *All maximal ideals of* D *have infinite* σ*-orbit.*

Then every homogeneous prime ideal of A *is* $J(\mathfrak{m})$ *for some* $\mathfrak{m} \in \operatorname{max\,spec} D$.

Proof: We just need to check that the hypotheses of Theorem 60 hold in the present situation. Hypothesis 61 implies that D is noetherian, and so A is noetherian by Proposition 4. By Proposition 69, the set $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\}$ is finite.

Fix $\mathfrak{m} \in \mathscr{M}$ and let $B = \{i \in \mathbb{Z} \mid \sigma^i(z) \in \mathfrak{m}\}$. Define a function $f : B \to \mathscr{M}$ by $f(i) = \sigma^{-i+1}(\mathfrak{m})$. If $i \in B$ is not a minimum element of B, then $\sigma^i(z) \in \mathfrak{m}$ and $\sigma^{i-n}(z) \in \mathfrak{m}$ for some $n > 0$. We can rewrite this conclusion as follows: $\sigma(z) \in f(i)$ and $\sigma^{-n+1}(z) \in f(i)$ for some $n > 0$. In other words, if $i \in B$ is not a minimum element of B then $f(i) \in \mathscr{M}'_{\mathrm{II}}$. Since f is an injection, we have obtained a bound $|B| \leq |\mathscr{M}'_{\mathrm{II}}| + 1$ on the cardinality of B. By Proposition 63, it follows that there is a uniform finite upper bound on the length of the $M_{\mathfrak{m}}$, for $\mathfrak{m} \in \mathscr{M}$. Therefore Theorem 60 applies to the present situation. ■

One hypothesis of Theorem 70 becomes automatic for a GWA over a domain of Krull dimension 1:

Corollary 71: *Assume in addition to Hypothesis 61 that* D *is a domain of Krull dimension* 1*, that* $z \neq 0$*, and that all maximal ideals of* D *have infinite* σ*-orbit. Then* $\operatorname{gr-spec}(A) = \{0\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{II}\}$.

Proof: Since D is a noetherian domain of Krull dimension 1, any infinite intersection of maximal ideals of D is 0. In other words, infinite subsets of $\max\operatorname{spec} D$ are dense. So whenever $\operatorname{Supp} V_{\mathfrak{m}}$ is infinite, its closure is all of $\max\operatorname{spec} D$. By Proposition 36, the set $\operatorname{Supp} V_{\mathfrak{m}}$ is infinite whenever $\mathfrak{m} \in \mathscr{M}_{\mathrm{I}}$. Thus $\{\overline{\operatorname{Supp} V_{\mathfrak{m}}} \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\} = \{\max\operatorname{spec} D\}$ is finite. For every $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$, we have $\sigma(z) \in \mathfrak{m}$. Since $\sigma(z) \neq 0$, it cannot be that $\sigma(z)$ is in infinitely many distinct maximal ideals of D. Thus $\mathscr{M}'_{\mathrm{II}}$ must be finite.

Applying Theorem 70, every homogeneous prime ideal of A is $J(\mathfrak{m})$ for some $\mathfrak{m} \in \mathscr{M}$. Using Proposition 64 we have

$$\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\} = \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{II}}\} = \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}\}.$$

If $\mathfrak{m} \in \mathscr{M}_{\mathrm{I}}$ then, using Proposition 66 and the fact that infinite intersections of maximal ideals of D are 0, we get that $J(\mathfrak{m}) = 0$. So $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\} = \{0\}$. We conclude that $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\} = \{0\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}\}$. ∎

2.6.3 Contraction to σ-Primes

The hypotheses of Theorem 70 are fairly specific, and this seems at first glance to severely restrict its applicability. In practice, Theorem 70 often does not apply directly to a given GWA. Instead, it applies to quotients and/or localizations of GWAs that make the base ring sufficiently simple. This suggests a general technique: use quotients and localizations to partition the prime spectrum of a GWA into pieces that look like prime spectra of GWAs to which Theorem 70 applies. This technique vastly widens the range of applicability for Theorem 70. We saw in Propositions 12 and 22 that the "nicest" quotients and localizations are the ones by σ-stable ideals and σ-stable multiplicative systems.

Let D be a ring and let σ be an automorphism of D. An ideal I of D is called a *σ-ideal* if $\sigma(I) \subseteq I$. By directly using the ascending chain condition on ideals of D, it is easy to show that $\sigma(I) = I$ when I is a σ-ideal and D is noetherian (see [17, Section 2]). In other words, the notion of σ-ideal is equivalent to the notion of σ^{-1}-ideal, in a noetherian ring. We say D is *σ-simple* if it has no proper nonzero σ-ideals. A *σ-prime* ideal is a proper σ-ideal P such that whenever $IJ \subseteq P$, where I and J are σ-ideals, we get $I \subseteq P$ or $J \subseteq P$. We denote by $\sigma\text{-spec}\,(D)$ the set of σ-prime ideals of D. The *σ-core* of an ideal I of D is the largest σ-ideal contained in I, denoted by $(I : \sigma)$. It is easy to see that $(I : \sigma)$ can be expressed as $I \cap \sigma^{-1}(I) \cap \sigma^{-2}(I) \cap \cdots$ or, if D is noetherian, as $I \cap \sigma(I) \cap \sigma^2(I) \cap \cdots$.

Proposition 72: *Let $A = D[x, y; \sigma, z]$ be a GWA with D noetherian and let P be a prime ideal of A. Then $(P \cap D : \sigma)$ is a σ-prime ideal of D.*

Proof: Observe that whenever I is a σ-ideal of D, its extension $\langle I \rangle$ to A is $\bigoplus_{m\in\mathbb{Z}} I v_m$.

Let $\mathfrak{p} = (P \cap D : \sigma)$. Suppose that I_1 and I_2 are σ-ideals of D and $I_1 I_2 \subseteq \mathfrak{p}$. Then

$$\langle I_1\rangle\langle I_2\rangle = \left(\bigoplus_{m\in\mathbb{Z}} I_1 v_m\right)\left(\bigoplus_{m\in\mathbb{Z}} I_2 v_m\right) = \left(\bigoplus_{m\in\mathbb{Z}} I_1 I_2 v_m\right) = \langle I_1 I_2\rangle \subseteq \langle\mathfrak{p}\rangle \subseteq P.$$

Note that in the second equality we needed to have $\sigma(I_i) = I_i$ for each i, and this works because D is noetherian. Now since P is prime, we have $\langle I_i\rangle \subseteq P$ for some $i \in \{1, 2\}$. Thus $I_i = (\langle I_i\rangle \cap D : \sigma) \subseteq (P \cap D : \sigma) = \mathfrak{p}$. We conclude that $\mathfrak{p}$ is σ-prime. ∎

For a GWA over a σ-simple base ring, one part of the hypotheses of Theorem 70 is automatic:

Corollary 73: *Assume in addition to Hypothesis 61 that D is noetherian and σ-simple. Assume that all maximal ideals of D have infinite σ-orbit and that the set $\mathscr{M}'_{II}$ is finite. Then* $\text{gr-spec}(A) = \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_I \cup \mathscr{M}'_{II}\}$ *and* $J(\mathfrak{m}) = 0$ *for* $\mathfrak{m} \in \mathscr{M}_I$.

Proof: Given any $\mathfrak{m} \in \mathscr{M}_{\mathrm{I}}$, the ideal $I(\operatorname{Supp} V_{\mathfrak{m}})$ is clearly σ-stable, given the description in Proposition 36. So $I(\operatorname{Supp} V_{\mathfrak{m}}) = 0$ and therefore $\overline{\operatorname{Supp} V_{\mathfrak{m}}} = \max\operatorname{spec} D$. This ensures that $\{\overline{\operatorname{Supp} V_{\mathfrak{m}}} \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}}\}$ is finite, which means we can apply Theorem 70 to get that gr-spec$(A) = \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\}$. We have $\{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}\} = \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}_{\mathrm{I}} \cup \mathscr{M}'_{\mathrm{II}}\}$ due to Proposition 64. If $\mathfrak{m} \in \mathscr{M}_{\mathrm{I}}$, then we see from Proposition 66 that $J(\mathfrak{m})_m$ is a proper σ-ideal for all $m \in \mathbb{Z}$. This forces $J(\mathfrak{m}) = 0$. ■

2.7 Poisson GWAs

Definition 74: A *Poisson k-algebra* is an algebra equipped with a Lie bracket that is also a biderivation (see Definition 150 in the appendix). The Lie bracket of a Poisson algebra is called a *Poisson bracket*.

The *semiclassical limit* of a quantum algebra is the commutative Poisson algebra one obtains by "setting q equal to 1." More precisely:

Definition 75: Let $\mathbb{F}$ be a subring of the rational function field $k(\tau)$ that contains the polynomial ring $k[\tau]$, and let A be an $\mathbb{F}$-algebra. Assume that $\tau - 1$ is regular in A and that $A/\langle \tau - 1\rangle$ is commutative. Then the *semiclassical limit* A_1 of A is defined to be the commutative k-algebra $A/\langle \tau - 1\rangle$ with Poisson bracket given by

$$\{\overline{a}, \overline{b}\} = \overline{\left(\frac{ab - ba}{\tau - 1}\right)},$$

where $\overline{a}$ denotes the coset of any $a \in A$ in $A/\langle \tau - 1\rangle$.

A GWA $R[x, y; \sigma, z]$ over a commutative R is commutative when the automorphism σ is chosen to be the identity map on R. As an algebra, this is what a Poisson GWA ought to be. What Poisson bracket should it have? We will know we have the correct construction if the following happens: whenever a GWA built from that data (R, σ, z) has a semiclassical limit, that semiclassical limit is a Poisson GWA that can be built from data that is suitably derived from (R, σ, z). This section is devoted to defining Poisson GWAs and then showing that the construction is correct. We begin by finding a convenient way to express commutative ("trivial") GWAs:

Proposition 76: *The trivial GWA $R[x, y; id, z]$ is isomorphic as an R-ring to $R[x, y]/\langle yx - z\rangle$. Specifically, there is an isomorphism that extends*

$$R \hookrightarrow R[x, y] \xrightarrow{quo} R[x, y]/\langle yx - z\rangle$$

to $R[x, y; id, z]$ in such a way that maps $x \mapsto x + \langle yx - z\rangle$ and $y \mapsto y + \langle yx - z\rangle$.

Proof: Define the specified homomorphism $R[x, y; \mathrm{id}, z] \to R[x, y]/\langle yx - z\rangle$ by checking that the GWA relations (1) hold where needed, and also define its inverse in the obvious way, checking that $yx - z$ is sent to zero. ■

Next we show that the Poisson bracket we will need exists.

Definition 77: Let R be a commutative Poisson k-algebra. A derivation $\alpha : R \to R$ is called a *Poisson derivation* if it satisfies

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$$

for all $a, b \in R$.

Lemma 78: *Let R be a commutative Poisson k-algebra, let $z \in R$, and define $W = R[x, y; id, z]$. Let $\alpha : R \to R$ be any function. The following are equivalent:*

1. *The map α is a Poisson derivation of R, and z is a Poisson central element of R.*
2. *There is a (unique) Poisson bracket $\{-,-\}$ for W that extends the one for R and has*
   - $\{x, r\} = \alpha(r)x$ *for* $r \in R$,
   - $\{r, y\} = \alpha(r)y$ *for* $r \in R$,
   - $\{x, y\} = \alpha(z)$.

Proof: $2 \Rightarrow 1$: Assume 2. For any $r \in R$, we have

$$\{z, r\} = \{yx, r\} = y\{x, r\} + x\{y, r\} = \alpha(r)xy - \alpha(r)xy = 0,$$

so z is Poisson central in R. Recall from Proposition 4(1) that in W we have $rx = 0 \Rightarrow r = 0$ for $r \in R$. We can use this to show that α is a Poisson derivation. For $a, b \in R$ and $c \in k$ we have

$$\alpha(ca + b)x = \{x, ca + b\} = c\{x, a\} + \{x, b\} = (c\alpha(a) + \alpha(b))x,$$

so α is k-linear. For $a, b \in R$ we have

$$\alpha(ab)x = \{x, ab\} = a\{x, b\} + b\{x, a\} = (a\alpha(b) + b\alpha(a))x$$

and

$$\begin{aligned}
\alpha(\{a, b\})x &= \{x, \{a, b\}\} = \{a, \{x, b\}\} + \{\{x, a\}, b\} = \{a, \alpha(b)x\} + \{\alpha(a)x, b\} \\
&= \{a, \alpha(b)\}x + \{a, x\}\alpha(b) + \{\alpha(a), b\}x + \{x, b\}\alpha(a) \\
&= \{a, \alpha(b)\}x - \alpha(a)\alpha(b)x + \{\alpha(a), b\}x + \alpha(a)\alpha(b)x \\
&= (\{a, \alpha(b)\} + \{\alpha(a), b\})x,
\end{aligned}$$

so α is a Poisson derivation.

$1 \Rightarrow 2$: Assume 1. By Proposition 76, we may work with $R[x, y]/\langle yx - z\rangle$ when constructing the required Poisson bracket. The construction will automatically be unique by Proposition 152.

Let $\{-,-\}_1 : R \times R \to R[x, y]$ be the Poisson bracket of R, viewed as a biderivation into the R-module $R[x, y]$. By Proposition 154, the biderivation $\{-,-\}_1$ has a unique extension $\{-,-\}_2 : R[x] \times R[x] \to R[x, y]$ satisfying the prescription

$$\{x, -\}_2 = \alpha(-)x, \qquad \{-, x\}_2 = -\alpha(-)x, \qquad \text{and} \qquad \{x, x\}_2 = 0.$$

(Note that since α is a derivation, the mapping $R \xrightarrow{\alpha(-)x} R[x, y]$ and its negative are derivations, a fact we needed in order to apply Proposition 154.) Define the derivation $\delta_y^l : R[x] \to R[x, y]$ to be the extension of $R \xrightarrow{-\alpha(-)y} R[x, y]$ that sends x to $-\alpha(z)$ (using Proposition 148). Similarly, define the derivation $\delta_y^r : R[x] \to R[x, y]$ to be the extension of $R \xrightarrow{\alpha(-)y} R[x, y]$ that sends x to $\alpha(z)$. Now use Proposition 154 again; the biderivation $\{-,-\}_2$ has a unique extension $\{-,-\}_3 : R[x, y] \times R[x, y] \to R[x, y]$ satisfying the prescription

$$\{y, -\}_3 = \delta_y^l, \qquad \{-, y\}_3 = \delta_y^r, \qquad \text{and} \qquad \{y, y\}_3 = 0.$$

According to Proposition 156, in order to show that $\{-,-\}_3$ is a Poisson bracket it suffices to check the needed identities on a generating set such as $R \cup \{x, y\}$. For $a, b \in R$ we have

$$\begin{aligned}
&\{a, b\}_3 = \{a, b\}_1 = -\{b, a\}_1 = -\{b, a\}_3 \\
&\{x, a\}_3 = \{x, a\}_2 = \alpha(a)x = -\{a, x\}_2 = -\{a, x\}_3 \\
&\{y, a\}_3 = \delta_y^l(a) = -\alpha(a)y = -\delta_y^r(a) = -\{a, y\}_3 \\
&\{x, y\}_3 = \delta_y^r(x) = \alpha(z) = -\delta_y^l(x) = -\{y, x\}_3 \\
&\{x, x\}_3 = \{x, x\}_2 = 0 \\
&\{y, y\}_3 = 0,
\end{aligned}$$

so antisymmetry holds for elements of the generating set $R \cup \{x, y\}$. For $a, b, c \in R$ we have

$$\begin{aligned}
&\{a, \{b, c\}_3\}_3 + \{b, \{c, a\}_3\}_3 + \{c, \{a, b\}_3\}_3 \\
&\qquad = \{a, \{b, c\}_1\}_1 + \{b, \{c, a\}_1\}_1 + \{c, \{a, b\}_1\}_1 = 0 \\
&\{a, \{b, x\}_3\}_3 + \{b, \{x, a\}_3\}_3 + \{x, \{a, b\}_3\}_3 \\
&\qquad = \{a, -\alpha(b)x\}_3 + \{b, \alpha(a)x\}_3 + \{x, \{a, b\}_1\}_3 \\
&\qquad = -\alpha(b)\{a, x\}_3 + \{a, -\alpha(b)\}_3 x + \alpha(a)\{b, x\}_3 + \{b, \alpha(a)\}_3 x + \alpha(\{a, b\}_1)x \\
&\qquad = \alpha(b)\alpha(a)x - \{a, \alpha(b)\}_1 x - \alpha(a)\alpha(b)x \\
&\qquad\qquad - \{\alpha(a), b\}_1 x + \alpha(\{a, b\}_1)x \\
&\qquad = -\{a, \alpha(b)\}_1 x - \{\alpha(a), b\}_1 x + \alpha(\{a, b\}_1)x \\
&\qquad = 0 \qquad\qquad\qquad\qquad \text{(because α is a Poisson derivation)} \\
&\{a, \{b, y\}_3\}_3 + \{b, \{y, a\}_3\}_3 + \{y, \{a, b\}_3\}_3 \\
&\qquad = \{a, \alpha(b)y\}_3 + \{b, -\alpha(a)y\}_3 + \{y, \{a, b\}_1\}_3 \\
&\qquad = \alpha(b)\{a, y\}_3 + \{a, \alpha(b)\}_3 y - \alpha(a)\{b, y\}_3 \\
&\qquad\qquad - \{b, \alpha(a)\}_3 y - \alpha(\{a, b\}_1)y \\
&\qquad = \alpha(b)\alpha(a)y + \{a, \alpha(b)\}_1 y - \alpha(a)\alpha(b)y \\
&\qquad\qquad + \{\alpha(a), b\}_1 y - \alpha(\{a, b\}_1)y \\
&\qquad = \{a, -\alpha(b)\}_1 y + \{\alpha(a), b\}_1 y - \alpha(\{a, b\}_1)y \\
&\qquad = 0 \qquad\qquad\qquad\qquad \text{(because α is a Poisson derivation)} \\
&\{a, \{x, y\}_3\}_3 + \{x, \{y, a\}_3\}_3 + \{y, \{a, x\}_3\}_3 \\
&\qquad = \{a, \alpha(z)\}_1 + \{x, -\alpha(a)y\}_3 + \{y, -\alpha(a)x\}_3 \\
&\qquad = \{a, \alpha(z)\}_1 - \{x, \alpha(a)\}_3 y - \{x, y\}_3 \alpha(a) \\
&\qquad\qquad\qquad\qquad - \{y, \alpha(a)\}_3 x - \{y, x\}_3 \alpha(a) \\
&\qquad = \{a, \alpha(z)\}_1 - \alpha^2(a)xy + \alpha^2(a)_3 xy \\
&\qquad = 0 \qquad\qquad\qquad\qquad \text{($\alpha(z)$ is Poisson central by Proposition 157)} \\
&\{a, \{y, x\}_3\}_3 + \{y, \{x, a\}_3\}_3 + \{x, \{a, y\}_3\}_3 \\
&\qquad = -(\{a, \{x, y\}_3\}_3 + \{y, \{a, x\}_3\}_3 + \{x, \{y, a\}_3\}_3) \\
&\qquad = 0 \qquad\qquad\qquad\qquad \text{(by the above case).}
\end{aligned}$$

This covers all the needed cases in order to verify the Jacobi identity on the generating set $R \cup \{x, y\}$; all remaining cases are either already covered by the above due to the cyclic symmetry of the Jacobi identity, or they they involve a repeated item, in which case the Jacobi identity reduces to an antisymmetry statement. Thus $\{-, -\}_3$ is a Poisson bracket for $R[x, y]$.

We now make the observation that $yx - z$ is Poisson central in $R[x, y]$; we have

$$\begin{aligned}
\{yx - z, r\}_3 &= \{y, r\}_3 x + \{x, r\}_3 y - \{z, r\}_3 = -\alpha(r)yx + \alpha(r)xy = 0 \\
\{yx - z, x\}_3 &= \{y, x\}_3 x - \{z, x\}_3 = -\alpha(z)x + \alpha(z)x = 0 \\
\{yx - z, y\}_3 &= \{x, y\}_3 y - \{z, y\}_3 = \alpha(z)y - \alpha(z)y = 0
\end{aligned}$$

for $r \in R$, so by Proposition 147 we conclude that $\{yx - z, -\}_3$ equals the zero derivation. The fact that $yx - z$ is Poisson central makes it easy to check that

$$\{I, R[x, y]\}_3 + \{R[x, y], I\}_3 \subseteq \langle yx - z \rangle,$$

where $I := \langle yx - z \rangle$. Finally, we can apply Proposition 155 to obtain an induced biderivation $\{-, -\} : R[x, y]/I \times R[x, y]/I \to R[x, y]/I$. This is clearly a Poisson bracket (since $\{-, -\}_3$ was) and it has the properties needed for assertion 2. ■

Now we can make our definition and prove that it operates properly.

Definition 79: Let R be a commutative Poisson k-algebra, let $z \in R$ be a Poisson central element, and let $\alpha : R \to R$ be a Poisson derivation. We denote by $R[x, y; \alpha, z]_P$ the Poisson k-algebra $R[x, y; \mathrm{id}, z]$ with the Poisson bracket from Lemma 78(2). We refer to $R[x, y; \alpha, z]_P$ as a *Poisson generalized Weyl algebra*, or *PGWA*.

When a GWA has a semiclassical limit, that semiclassical limit is a PGWA:

Theorem 80: *Let $\mathbb{F}$ be a subring of the rational function field $k(\tau)$ that contains the polynomial ring $k[\tau]$, let R be an $\mathbb{F}$-algebra, let $\sigma : R \to R$ be an $\mathbb{F}$-algebra automorphism, and let $z \in Z(R)$. Consider the GWA $W := R[x, y; \sigma, z]$. Assume that $\tau - 1$ is regular in W and that $W/\langle \tau - 1 \rangle$ is commutative. Then*

$$W_1 \cong R_1[x, y; \alpha, \overline{z}]_P$$

where $(-)_1$ denotes semiclassical limit, $\overline{z}$ denotes the image of z in R_1, and α is a Poisson derivation of R_1 given by

$$\alpha(\overline{r}) = \overline{(\tau - 1)^{-1}(\sigma(r) - r)}$$

for $r \in R$.

Proof: In order for the expression $R_1[x, y; \alpha, \overline{z}]_P$ to be defined, we need to show that R_1 exists (i.e. that the conditions of Definition 75 are met), that $\overline{z}$ is Poisson central in R_1, and that α exists and is a Poisson derivation of R_1. We use the notation $h := \tau - 1$ throughout this proof.

Since

$$\langle h \rangle_W = \bigoplus_{m \in \mathbb{Z}} \langle h \rangle_R v_m,$$

the homomorphism

$$R/\langle h \rangle_R \to W/\langle h \rangle_W \tag{30}$$

induced by $R \hookrightarrow W$ is an embedding. Hence $R/\langle h \rangle_R$ is commutative. And h is regular in R since it is regular in W. So the semiclassical limit R_1 exists. The element $\overline{z}$ of R_1 is Poisson central because $z \in R$ is central.

Since $W/\langle h \rangle_W$ is commutative,

$$(\sigma(r) - r)x = [x, r] \in \langle h \rangle_W = \bigoplus_{m \in \mathbb{Z}} \langle h \rangle_R v_m$$

for $r \in R$. It follows that

$$\sigma(r) - r \in \langle h \rangle_R \tag{31}$$

for $r \in R$, so there is a well-defined $\mathbb{F}$-linear map $R \to R_1$ given by $r \mapsto \overline{(h^{-1}(\sigma(r) - r))}$. Since this map sends any hr to $\overline{\sigma(r) - r} = 0$, it induces the desired $\alpha : R_1 \to R_1$. The k-linear map α is a derivation:

$$\begin{aligned}
\alpha(\overline{a}\overline{b}) - \overline{a}\alpha(\overline{b}) - \overline{b}\alpha(\overline{a}) &= \overline{h^{-1}(\sigma(ab) - ab)} - \overline{h^{-1}a(\sigma(b) - b)} - \overline{h^{-1}(\sigma(a) - a)b} \\
&= \overline{h^{-1}(\sigma(ab) - a\sigma(b) - \sigma(a)b + ab)} \\
&= \overline{h^{-1}(\sigma(a) - a)(\sigma(b) - b)} \\
&= 0 \qquad \text{(because } h^2 \text{ divides } (\sigma(a) - a)(\sigma(b) - b) \text{ due to (31))}
\end{aligned}$$

for $a, b \in R$. And further α is a Poisson derivation:

$$\begin{aligned}
\alpha(\{\overline{a}, \overline{b}\}) - \{\alpha(\overline{a}), \overline{b}\} - \{\overline{a}, \alpha(\overline{b})\} \\
&= \overline{h^{-1}(\sigma(h^{-1}[a,b]) - h^{-1}[a,b])} - \overline{h^{-1}[h^{-1}(\sigma(a)-a), b]} - \overline{h^{-1}[a, h^{-1}(\sigma(b)-b)]} \\
&= \overline{h^{-2}(\sigma([a,b]) - [a,b] - [\sigma(a)-a, b] - [a, \sigma(b)-b])} \\
&= \overline{h^{-2}([\sigma(a), \sigma(b)] + [a,b] - [\sigma(a), b] - [a, \sigma(b)])} \\
&= \overline{h^{-2}[\sigma(a)-a, \sigma(b)-b]} \\
&= 0 \qquad (\text{because } h^3 \mid [\sigma(a)-a, \sigma(b)-b] \text{ due to (31)} \\
&\qquad\qquad \text{and the fact that } W/\langle h \rangle \text{ is commutative})
\end{aligned}$$

for $a, b \in R$.

Now Lemma 78 provides us with a Poisson k-algebra $R_1[x, y; \alpha, \overline{z}]_P$. We define a homomorphism $\psi : W_1 \to R_1[x, y; \alpha, \overline{z}]_P$. Start with

$$R \xrightarrow{quo} R_1 \hookrightarrow R_1[x, y; \alpha, \overline{z}]_P,$$

and extend it to $R[x, y; \sigma, z]$ by sending $x \mapsto x$ and $y \mapsto y$; this works because the needed GWA relations (1) hold inside $R_1[x, y; \alpha, \overline{z}]_P$:

$$\begin{array}{lll}
yx = \overline{z} & xy = \overline{z} = \overline{\sigma(z)} & \\
x\overline{r} = \overline{r}x = \overline{\sigma(r)}x & y\overline{\sigma(r)} = \overline{\sigma(r)}y = \overline{r}y & \forall\, r \in R
\end{array}$$

(here we have made use of (31), i.e. $\overline{\sigma(r)} = \overline{r}$ for $r \in R$). Since $h \mapsto 0$, we may define $\psi : W_1 \to R[x, y; \alpha, \overline{z}]_P$ to be the induced homomorphism.

Now we define a homomorphism $\phi : R[x, y; \alpha, \overline{z}]_P \to W_1$ in the opposite direction. To make things simpler, we identify $R[x, y; \alpha, \overline{z}]_P$ with $R[x, y]/\langle yx - z \rangle$, as in Proposition 76. Start with the embedding $R_1 \hookrightarrow W_1$ from (30) and extend it to $R_1[x, y]$ by sending $x \mapsto \overline{x}$ and $y \mapsto \overline{y}$; this works because W_1 is commutative. Since $yx - \overline{z} \mapsto 0$, we may define $\phi : R[x, y; \alpha, \overline{z}]_P \to W_1$ to be the induced homomorphism.

It is clear that ϕ and ψ are mutually inverse, so ϕ is an isomorphism of k-algebras. It remains only to show that the Poisson bracket is preserved. For clarity of notation we write out cosets fully in the calculation that follows. We have for $r \in R$:

$$\begin{aligned}
\{\phi(x), \phi(r + \langle h \rangle_R)\} &= \{x + \langle h \rangle_W, r + \langle h \rangle_W\} = \tfrac{xr - rx}{h} + \langle h \rangle_W = \tfrac{\sigma(r) - r}{h} x + \langle h \rangle_W \\
&= \phi(\tfrac{\sigma(r)-r}{h} + \langle h \rangle_R)\phi(x) = \phi(\alpha(r + \langle h \rangle_R)x) = \phi(\{x, r + \langle h \rangle_R\}) \\
\{\phi(r + \langle h \rangle_R), \phi(y)\} &= \{r + \langle h \rangle_W, y + \langle h \rangle_W\} = \tfrac{ry - yr}{h} + \langle h \rangle_W = y\tfrac{\sigma(r)-r}{h} + \langle h \rangle_W \\
&= \phi(y)\phi(\tfrac{\sigma(r)-r}{h} + \langle h \rangle_R) = \phi(\alpha(r + \langle h \rangle_R)y) = \phi(\{r + \langle h \rangle_R, y\}) \\
\{\phi(x), \phi(y)\} &= \{x + \langle h \rangle_W, y + \langle h \rangle_W\} = \tfrac{xy - yx}{h} + \langle h \rangle_W = \tfrac{\sigma(z) - z}{h} + \langle h \rangle_W \\
&= \phi(\tfrac{\sigma(z)-z}{h} + \langle h \rangle_R) = \phi(\alpha(z + \langle h \rangle_R)) = \phi(\{x, y\}).
\end{aligned}$$

Thus (by Proposition 153) ϕ is an isomorphism of Poisson k-algebras. ∎

3 Applications

Our main and most completely worked out example is the 2×2 reflection equation algebra. The other examples that appear below were mainly used to further test and demonstrate the theory presented in section 2.6. Before jumping into the examples, we give here a few observations that will help us to apply that theory.

The following linear algebra observation will be useful to reference later on:

Proposition 81: *Let A be a $\mathbb{Z}$-graded k-algebra. Suppose there is some k-linear operator $L : A \to A$ such that the graded components of A are precisely the distinct eigenspaces for L. Any two-sided ideal of A that is L-invariant is homogeneous.*

Next we verify that maximal ideals have infinite orbit with respect to automorphisms that scale something (by a non-root-of-unity) or shift something (in characteristic zero). These Propositions would be more obvious if k were algebraically closed.

Proposition 82: *Let D be a commutative affine k-algebra, let u be an element of D, and let σ an automorphism of D such that $\sigma(u) = \alpha u$, where $\alpha \in k$ is not a root of unity. If $\mathfrak{m} \in \operatorname{max\,spec} D$ has finite σ-orbit, then $u \in \mathfrak{m}$. Thus, if u is a unit then every maximal ideal of D has infinite σ-orbit.*

Proof: Let $\mathfrak{m} \in \operatorname{max\,spec} D$ such that $\sigma^n(\mathfrak{m}) = \mathfrak{m}$, where $n > 0$. An automorphism τ of the field $K := D/\mathfrak{m}$ is induced by σ^n. The field extension K is finite over k. Let $\overline{u}$ be the image of u in K, and let $f \in k[x]$ be the minimal polynomial of $\overline{u}$ over k. Since $\overline{u}, \tau(\overline{u}), \tau^2(\overline{u}), \ldots$ are roots of f, there must be some $i > 0$ such that $\tau^i(\overline{u}) = \overline{u}$. Then $(\alpha^{ni} - 1)\overline{u} = 0$ while $ni > 0$, so $\overline{u} = 0$. ■

Proposition 83: *Assume k has characteristic 0. Let D be a commutative affine k-algebra, let h be an element of D, and let σ an automorphism of D such that $\sigma(h) = h + \alpha$, where $\alpha \in k^\times$. Then every maximal ideal of D has infinite σ-orbit.*

Proof: Let $\mathfrak{m} \in \operatorname{max\,spec} D$ such that $\sigma^n(\mathfrak{m}) = \mathfrak{m}$, where $n > 0$. An automorphism τ of the field $K := D/\mathfrak{m}$ is induced by σ^n. The field extension K is finite over k. Let $\overline{h}$ be the image of h in K, and let $f \in k[x]$ be the minimal polynomial of $\overline{h}$ over k. Since $\overline{h}, \tau(\overline{h}), \tau^2(\overline{h}), \ldots$ are roots of f, there must be some $i > 0$ such that $\tau^i(\overline{h}) = \overline{h}$. Then we have the contradiction $(ni)\alpha = 0$, with $ni > 0$. ■

In order to apply Theorem 70, one must verify that the commutative base ring is a Jacobson ring. One often uses localization to make Theorem 70 applicable, so here is one tool for dealing with the localizations that come up in this work. The proof of the following proposition is adapted from the ideas on [40, page 157].

Proposition 84: *Let $R = \bigoplus_{m \in \mathbb{Z}^n} R_m$ be a $\mathbb{Z}^n$-graded commutative ring, where $n \geq 1$. Let T be the localization of R at the set of nonzero homogeneous elements. Then T is an affine algebra over some field.*

Proof: For $m \in \mathbb{Z}^n$, let T_m denote the set of $r/s \in T$ for which $r, s \in R$ are homogeneous with $\deg(r) - \deg(s) = m$. It is clear that each T_m is an additive subgroup of T and that $T_m T_{m'} \subseteq T_{m+m'}$ for $m, m' \in \mathbb{Z}^m$. Thus $\sum_{m \in \mathbb{Z}^n} T_m$ is a subring of T. It is easy to see that $\sum_{m \in \mathbb{Z}^n} T_m$ must be the entire localization T. The set $\mathcal{I} := \{m \in \mathbb{Z}^n \mid T_m \neq 0\}$ is a subgroup of $\mathbb{Z}^n$. Therefore $\mathcal{I}$ has a basis $m_1, \ldots, m_t$. Choose a nonzero element y_i of each T_{m_i}. For any $m \in \mathbb{Z}^n$ and any nonzero $a \in T_m$, we may write m as $c_1 m_1 + \cdots + c_t m_t$ for some $c_1, \ldots, c_t \in \mathbb{Z}$ and then we have $a = (a y_1^{-c_1} \cdots y_t^{-c_t}) y_1^{c_1} \cdots y_t^{c_t} \in T_0 y_1^{c_1} \cdots y_t^{c_t}$. Thus T is generated as a ring over T_0 by the elements $y_1^{\pm 1}, \ldots, y_t^{\pm 1}$. This makes T a commutative affine algebra over the field T_0. ■

Surprisingly often, we will find ourselves in the following specific situation:

Proposition 85: *Let $A = D[x, y; \sigma, z]$ be a GWA over a commutative noetherian k-algebra D. Assume that D is a direct sum of σ-eigenspaces, and further that D has a $\mathbb{Z}^n$-grading for some $n \geq 1$ such that the graded components coincide with the σ-eigenspaces. Let $\mathfrak{p} \in \sigma\text{-spec}(D)$ and let $D' = D/\mathfrak{p}$. Let $\mathcal{E}$ be the set of nonzero σ-eigenvectors in D'. Let $D'' = D'\mathcal{E}^{-1}$. Using Propositions 12 and 22, identify*

$A' := A/\langle\mathfrak{p}\rangle$ with $D'[x,y;\sigma,z]$ and identify $A'' := A'\mathcal{E}^{-1}$ with $D''[x,y;\sigma,z]$. Then:

1. *D'' is a σ-simple affine algebra over some field.*
2. *Pullback along the localization and quotient maps $A \to A' \to A''$ defines a bijection*

$$\mathrm{spec}(A'') \to \{P \in \mathrm{spec}(A) \mid (P \cap D : \sigma) = \mathfrak{p}\}.$$

Proof: The assumptions imply that "σ-ideal" and "homogeneous ideal" are the same notion in D. Hence D' inherits the $\mathbb{Z}^n$-grading from D, and $\mathcal{E}$ can be described as its set of nonzero homogeneous elements. Proposition 84 then tells us that D'' is an affine algebra over some field. If I is a nonzero σ-ideal of D'', then its contraction I^c to D' is also nonzero and σ-invariant. In other words, I^c is a nonzero homogeneous ideal. It must then contain an element of $\mathcal{E}$, so I must have been the unit ideal. Therefore D'' is σ-simple.

Pullback along the quotient map $A \to A'$ provides a bijection

$$\{P \in \mathrm{spec}(A') \mid (P \cap D' : \sigma) = 0\} \to \{P \in \mathrm{spec}(A) \mid (P \cap D : \sigma) = \mathfrak{p}\}.$$

We claim that an ideal I of D' has $(I : \sigma) = 0$ if and only if I is disjoint from $\mathcal{E}$. Indeed, if I contains a nonzero σ-eigenvector x then one has $x \in (I : \sigma) \neq 0$. And if $(I : \sigma) \neq 0$, then I contains the nonzero homogeneous ideal $(I : \sigma)$ and therefore contains an element of $\mathcal{E}$. It follows that given a $P \in \mathrm{spec}(A')$, the conditions $(P \cap D' : \sigma) = 0$ and $P \cap \mathcal{E} = \emptyset$ are equivalent. Thus, pullback along the localization map $A' \to A''$ provides a bijection

$$\mathrm{spec}(A'') \to \{P \in \mathrm{spec}(A') \mid (P \cap D' : \sigma) = 0\}.$$

Putting the two bijections together gives the one claimed by the Proposition. ■

3.1 The 2×2 Reflection Equation Algebra

We begin with an origin story and definition for the algebra $\mathcal{A}_q(\mathrm{M}_2)$. Throughout this section, $q \in k^\times$ is not a root of unity.

Let n be a positive integer. Consider the action by right conjugation of the algebraic group $\mathrm{GL}_n(k)$ of invertible $n \times n$ matrices on the space $\mathrm{M}_n(k)$ of all $n \times n$ matrices:

$$M \xmapsto{g \in \mathrm{GL}_n(k)} g^{-1}Mg.$$

At the level of coordinate rings, the action map becomes an algebra homomorphism

$$\mathcal{O}(\mathrm{M}_n) \to \mathcal{O}(\mathrm{M}_n) \otimes \mathcal{O}(\mathrm{GL}_n), \tag{32}$$

where we have dropped mention of the base field k to simplify notation. This gives $\mathcal{O}(\mathrm{M}_n)$ the structure of a comodule-algebra over the Hopf algebra $\mathcal{O}(\mathrm{GL}_n)$. We shall consider what happens when this picture is carried into a quantum algebra setting. The construction of [37] yields a noncommutative deformation $\mathcal{O}_q(\mathrm{M}_n)$ of $\mathcal{O}(\mathrm{M}_n)$, using the the R-matrix

$$R^{ik}_{jl} = \begin{cases} q & \text{if } i = j = k = l \\ 1 & \text{if } i = j,\ k = l, \text{ and } i \neq k \\ q - q^{-1} & \text{if } i > j,\ i = l, \text{ and } j = k \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

More precisely, the k-algebra $\mathcal{O}_q(\mathrm{M}_n)$ has a presentation with n^2 generators $\{t^i_j \mid 0 \leq i, j \leq n\}$ and the relations

$$R^{ik}_{ab} t^a_j t^b_l = R^{ab}_{jl} t^k_b t^i_a.$$

Here, and throughout this section, we have adopted a convention in which a repeated index in an expression indicates an implicit summation from 1 to n. The algebra $\mathcal{O}_q(\mathrm{M}_n)$ is a bialgebra in a way that matches the comultiplication on $\mathcal{O}(\mathrm{M}_n)$ induced by matrix multiplication in M_n:

$$\Delta(t^i_j) = t^i_k \otimes t^k_j$$
$$\epsilon(t^i_j) = \delta^i_j,$$

where δ^i_j is the Kronecker delta, which equals $1 \in k$ when $i = j$ and otherwise equals $0 \in k$. Inverting a suitable central determinant-like element in $\mathcal{O}_q(\mathrm{M}_n)$ yields a noncommutative deformation $\mathcal{O}_q(\mathrm{GL}_n)$ of $\mathcal{O}(\mathrm{GL}_n)$; see [8, I.2.4] for example. The bialgebra $\mathcal{O}_q(\mathrm{GL}_n)$ is a Hopf algebra; we use S to denote its antipode.

One may attempt to mimic the map of (32) for $\mathcal{O}_q(\mathrm{M}_n)$ with the hope of making this algebra a comodule-algebra over $\mathcal{O}_q(\mathrm{GL}_n)$,

$$\begin{array}{ccc} \mathcal{O}_q(\mathrm{M}_n) & \to & \mathcal{O}_q(\mathrm{M}_n) \otimes \mathcal{O}_q(\mathrm{GL}_n) \\ t^i_j & \mapsto & t^k_l \otimes S(t^i_k)t^l_j, \end{array} \tag{34}$$

but such a prescription yields only a coaction map and not an algebra homomorphism. The remedy is to replace the $\mathcal{O}_q(\mathrm{GL}_n)$-comodule $\mathcal{O}_q(\mathrm{M}_n)$ by a different noncommutative deformation of $\mathcal{O}(\mathrm{M}_n)$. The needed construction is provided by the transmutation theory of Majid, presented in [31]; it is a k-algebra $\mathcal{A}_q(\mathrm{M}_n)$ with n^2 generators $\{u^i_j \mid 0 \leq i, j \leq n\}$ and the relations

$$R^{li}_{mn} R^{pm}_{qr} u^k_l u^n_p \;=\; R^{ki}_{ml} R^{nm}_{qp} u^l_n u^p_r, \tag{35}$$

where the R-matrix is still (33), the same one used to build $\mathcal{O}_q(\mathrm{M}_n)$. Replacing (34) with

$$\begin{array}{ccc} \mathcal{A}_q(\mathrm{M}_n) & \to & \mathcal{A}_q(\mathrm{M}_n) \otimes \mathcal{O}_q(\mathrm{GL}_n) \\ u^i_j & \mapsto & u^k_l \otimes S(t^i_k)t^l_j \end{array}$$

does give an algebra homomorphism, making $\mathcal{A}_q(\mathrm{M}_n)$ a comodule-algebra over $\mathcal{O}_q(\mathrm{GL}_n)$ and providing a more suitable "quantization" of (32). The algebra $\mathcal{A}_q(\mathrm{M}_n)$ is referred to as a *braided matrix algebra* by Majid, and as a *reflection equation algebra* elsewhere in the literature.

We shall focus on the case $n = 2$, the 2×2 reflection equation algebra, denoted throughout this section by $\mathcal{A} := \mathcal{A}_q(\mathrm{M}_2)$. Using u_{ij} in place of u^i_j, the algebra $\mathcal{A}$ is generated by u_{ij} for $i, j \in \{1, 2\}$ with the relations given in (35), which simplify to:

$$\begin{array}{ll} u_{11}u_{22} = u_{22}u_{11} & \\ u_{11}u_{12} = u_{12}(u_{11} + (q^{-2} - 1)u_{22}) & u_{21}u_{11} = (u_{11} + (q^{-2} - 1)u_{22})u_{21} \\ u_{22}u_{12} = q^2 u_{12}u_{22} & u_{21}u_{22} = q^2 u_{22}u_{21} \\ u_{21}u_{12} - u_{12}u_{21} = (q^{-2} - 1)u_{22}(u_{22} - u_{11}). & \end{array} \tag{36}$$

Observe that u_{12} and u_{21} normalize the subalgebra generated by u_{11} and u_{22}, and they do so via inverse automorphisms of that subalgebra. This suggests that $\mathcal{A}$ is a GWA.

Brief History The "reflection equation" (35) was first introduced by Cherednik in his study [11] of factorizable scattering on a half-line, and reflection equation algebras later emerged from Majid's transmutation theory in [30]. In [28], Kulish and Sklyanin prove several things about $\mathcal{A} = \mathcal{A}_q(\mathrm{M}_2)$. They show that $\mathcal{A}$ has a k-basis consisting of monomials in the generators u_{ij}. They compute the center of $\mathcal{A}$. They find a determinant-like element of $\mathcal{A}$ and they show that inverting u_{22} and setting the determinant-like element equal to 1 yields $U_q(\mathfrak{sl}_2)$, and they note that this can be used to pull back representations of $U_q(\mathfrak{sl}_2)$ to representations of $\mathcal{A}$. (We shall see in this thesis that all irreducible representations that are not annihilated by u_{22} arise in this way.) Domokos and Lenagan address $\mathcal{A}_q(\mathrm{M}_n)$ for general n in [13]. They show that $\mathcal{A}_q(\mathrm{M}_n)$ is a noetherian domain, and that it has a k-basis consisting of monomials in the generators u_{ij}.

3.1.1 First Results

Define an automorphism σ of the polynomial ring $k[u_{22}, u_{11}, z]$ by

$$\begin{array}{rcl} \sigma(u_{22}) & = & q^2 u_{22} \\ \sigma(u_{11}) & = & u_{11} + (q^{-2} - 1)u_{22} \\ \sigma(z) & = & z + (q^{-2} - 1)u_{22}(u_{22} - u_{11}). \end{array} \tag{37}$$

Proposition 86: *The algebra $\mathcal{A}$ is a GWA over the above polynomial algebra, with x being u_{21} and y being u_{12}:*

$$\mathcal{A} \cong k[u_{22}, u_{11}, z][x, y; \sigma, z].$$

Proof: This can be verified by defining mutually inverse homomorphisms in both directions using universal properties. One checks that the reflection equation relations (36) hold in the GWA, and that the GWA relations (1) hold in $\mathcal{A}$. ■

Proposition 87: *$\mathcal{A}$ is a noetherian domain of GK dimension* 4.

Proof: In [13, Proposition 3.1], polynormal sequences and Gröbner basis techniques are used to show that $\mathcal{A}_q(\mathrm{M}_n)$ is a noetherian domain for all n. Proposition 4 and Corollary 6 give an alternative way to see this for $\mathcal{A} = \mathcal{A}_q(\mathrm{M}_2)$.

It is also observed in [13] that the Hilbert series of $\mathcal{A}_q(\mathrm{M}_n)$ can be determined using [31, (7.37)]. One may deduce from the Hilbert series that the GK dimension of $\mathcal{A}_q(\mathrm{M}_n)$ is n^2. Theorem 28 gives an alternative way to see this for $\mathcal{A} = \mathcal{A}_q(\mathrm{M}_2)$, since σ is locally algebraic. ■

By a change of variables in $k[u_{22}, u_{11}, z]$ we can greatly simplify the expression of $\mathcal{A}$ as a GWA. Consider the change of variables:

$$\begin{array}{l} u = u_{22} \\ t = u_{11} + q^{-2}u_{22} \\ d = z - q^{-2}u_{11}u_{22} \end{array} \tag{38}$$

Now we have

$$\mathcal{A} \cong k[u, t, d][x, y; \sigma, z], \tag{39}$$

where $z = d + q^{-2}tu - q^{-4}u^2$ and

$$\begin{array}{rcl} \sigma(u) & = & q^2 u \\ \sigma(t) & = & t \\ \sigma(d) & = & d. \end{array} \tag{40}$$

The special elements t and d of $\mathcal{A}$ are, up to a scalar multiple, the quantum trace and quantum determinant explored in [30].

Since q is not a root of unity, σ has infinite order. We may therefore apply Proposition 8 to determine the center of $\mathcal{A}$:

Proposition 88: $Z(\mathcal{A}) = k[t, d]$.

This was also computed in [28], and a complete description of the center of $\mathcal{A}_q(\mathrm{M}_n)$ for arbitrary n is given in [23].

Using the fact that q is not a root of unity, the elements

$$\sigma^m(z) = d + q^{2m-2}tu - q^{4m-4}u^2$$

of $k[u,t,d]$, for $m \in \mathbb{Z}$, are pairwise coprime. This allows us to get at the normal elements of $\mathcal{A}$, which gives us a handle on its automorphism group:

Theorem 89: *The automorphism group of $\mathcal{A}$ is isomorphic to $(k^\times)^2$, with $(\alpha,\gamma) \in (k^\times)^2$ corresponding to the automorphism given by*

$$\begin{matrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{matrix} \quad \mapsto \quad \begin{matrix} \alpha\, u_{11} & \frac{\alpha}{\gamma}\, u_{12} \\ \alpha\gamma\, u_{21} & \alpha\, u_{22}\,. \end{matrix}$$

Proof: Let $\psi : \mathcal{A} \to \mathcal{A}$ be an automorphism. By Proposition 11, the nonzero normal elements of $\mathcal{A}$ are the σ-eigenvectors in $k[u,t,d]$. That is, they all have the form $u^i f(t,d)$ for some polynomial $f(t,d)$ and some $i \in \mathbb{Z}_{\geq 0}$. Since $k[u,t,d]$ is the linear span of such elements, it is preserved by ψ. Since u is normal, $\psi(u) = u^i f(t,d)$ for some i and f, and similarly $\psi^{-1}(u) = u^j g(t,d)$ for some j and g. Note that $k[t,d]$, being the center of $\mathcal{A}$, is also preserved by ψ. Therefore $u = \psi(\psi^{-1}(u)) = u^{ij} f^j \psi(g)$ implies that $i = 1$ and f is a unit. So $\psi(u) = \alpha u$ for some $\alpha \in k^\times$.

Observe that $\psi(x)u = \alpha^{-1}\psi(xu) = \alpha^{-1}q^2\psi(ux) = q^2 u\psi(x)$. Any $a \in \mathcal{A}$ with the property that $au = q^2 ua$ is a sum of homogeneous such a's, and a homogeneous such a is bv_m for some $b \in k[u,t,d]$ and some $m \in \mathbb{Z}$ such that

$$q^2 ubv_m = bv_m u = q^{2m} buv_m.$$

This equation requires that either $b = 0$ or $m = 1$. Therefore $\psi(x) = bx$ for some nonzero $b \in k[u,t,d]$. The same argument applies to ψ^{-1}, and it is easy to deduce from this that b must be a unit, i.e. $b \in k^\times$. Similarly, using the fact that $\psi(y)u = q^{-2}u\psi(y)$, we get that $\psi(y) = cy$ for some $c \in k^\times$.

For any $m > 0$, we have

$$\begin{aligned} b^{m-1}\psi(\sigma^m(z))x^{m-1} &= \psi(\sigma^m(z)x^{m-1}) \\ &= \psi(x^m y) \\ &= b^m c x^m y \\ &= b^m c \sigma^m(z) x^{m-1}. \end{aligned}$$

It follows that $\psi(\sigma^m(z)) = bc\sigma^m(z)$ for all $m > 0$. Considering that $\sigma^m(z) = d + q^{2m-2}tu - q^{4m-4}u^2$, the linear span of $\{\sigma(z), \sigma^2(z), \sigma^3(z)\}$, for instance, contains $\{d, tu, u^2\}$. So $bc\,u^2 = \psi(u^2) = \alpha^2 u^2$, i.e. $bc = \alpha^2$. And $bc\,tu = \psi(tu) = \alpha\psi(t)u$, so $\psi(t) = \alpha t$. And $bc\,d = \psi(d)$, so $\psi(d) = \alpha^2 d$. Letting $\gamma = b\alpha^{-1}$, so that $\psi(x) = (\alpha\gamma)x$ and $\psi(y) = (\alpha\gamma^{-1})y$, we see that ψ is the automorphism corresponding to (α,γ) in the theorem statement. One easily checks that there is such an automorphism for every $(\alpha,\gamma) \in (k^\times)^2$, and that composition of automorphisms corresponds to multiplication in $(k^\times)^2$. ■

3.1.2 Finite Dimensional Simple Modules

The finite dimensional simple modules over $\mathcal{A}$ come in two types: the ones annihilated by u_{22} and the ones on which u_{22} acts invertibly. This observation follows from the fact that since u_{22} is normal, its annihilator in any $\mathcal{A}$-module is a submodule. The former are modules over $\mathcal{A}/\langle u_{22}\rangle$, a three-variable polynomial ring. The latter are addressed by Theorem 34 given the GWA structure (39). They will turn out to be pullbacks of simple modules over $U_q(\mathfrak{sl}_2)$, the k-algebra defined in [8, I.3]. We proceed to apply Theorem 34 and state a classification.

Assume that k is algebraically closed. Let R denote the coefficient ring $k[u,t,d]$ of $\mathcal{A}$ as a GWA. Maximal ideals of R take the form $\mathfrak{m}(u_0,t_0,d_0) := \langle u - u_0, t - t_0, d - d_0\rangle$ for some scalars $u_0, t_0, d_0 \in k$. They get moved by σ^n to $\mathfrak{m}(q^{-2n}u_0, t_0, d_0)$ for $n \in \mathbb{Z}$, so $\mathfrak{m}(u_0,t_0,d_0)$ has infinite σ-orbit if and only if $u_0 \neq 0$. Therefore a finite dimensional simple left $\mathcal{A}$-module contains a simple R-submodule with annihilator having infinite σ-orbit if and only if $u = u_{22}$ acts nontrivially. Theorem 34 requires us to consider the condition $\sigma^{-n+1}(z), \sigma^{n'}(z) \in \mathfrak{m}(u_0, t_0, d_0)$ where $n, n' > 0$. Since

$$\sigma^{-n+1}(z) = d + q^{-2n}tu - q^{-4n}u^2$$
$$\sigma^{n'}(z) = d + q^{2n'-2}tu - q^{4n'-4}u^2,$$

a straightforward calculation shows that, as long as $u_0 \neq 0$, one has $\sigma^{-n+1}(z), \sigma^{n'}(z) \in \mathfrak{m}(u_0, t_0, d_0)$ if and only if

$$d_0 = -q^{2(n'-n-1)}u_0^2 \qquad t_0 = (q^{-2n} + q^{2(n'-1)})u_0. \tag{41}$$

Define for $u_0 \in k^\times$ and $t_0, d_0 \in k$ the left $\mathcal{A}$-module $M(u_0, t_0, d_0) := \mathcal{A}/(\mathcal{A}\mathfrak{m}(u_0, t_0, d_0))$. For all $i \in \mathbb{Z}$, let e_i denote the image of v_i in $M(u_0, t_0, d_0)$. Let $N(u_0, t_0, d_0)$ be the submodule $\bigoplus_{i \leq -n'} Re_i \oplus \bigoplus_{i \geq n} Re_i$, where $n > 0$ is chosen to be minimal such that $d_0 + q^{-2n}t_0u_0 - q^{-4n}u_0^2 = 0$ (or ∞ if this does not occur), and $n' > 0$ is chosen to be minimal such that $d_0 + q^{2n'-2}t_0u_0 - q^{4n'-4}u_0^2 = 0$ (or ∞ if this does not occur). We observed in the general setting (22) that this is the unique largest proper submodule of $M(u_0, d_0, t_0)$. Define $V(u_0, t_0, d_0)$ to be the simple left $\mathcal{A}$-module $M(u_0, t_0, d_0)/N(u_0, t_0, d_0)$. As an R-module, this is isomorphic to

$$\bigoplus_{-n' < i < n} R/\sigma^i(\mathfrak{m}(u_0, t_0, d_0)),$$

so it has dimension $n + n' - 1$ when n and n' are finite. Putting together our observations and applying Theorem 34, we have:

Theorem 90: *Assume that k is algebraically closed.*

1. *Let $u_0 \in k^\times$ and let $t_0, d_0 \in k$. The simple left $\mathcal{A}$-module $V(u_0, t_0, d_0)$ is finite dimensional if and only if there are $n, n' > 0$ such that (41) holds.*
2. *Let $n > 0$. Any n-dimensional simple left $\mathcal{A}$-module V that is not annihilated by $u = u_{22}$ is isomorphic to*
$$V_n(u_0) \ := \ V(u_0,\ t_0 = (q^{-2n} + 1)u_0,\ d_0 = -q^{-2n}u_0^2)$$
for a unique $u_0 \in k^\times$, namely the eigenvalue of u_{22} on $\mathrm{ann}_V(u_{12})$.

These simple modules are all pullbacks of simple $U_q(\mathfrak{sl}_2)$-modules along homomorphisms. Define, for each $\alpha \in k^\times$, an algebra homomorphism $\psi_\alpha : \mathcal{A} \to U_q(\mathfrak{sl}_2)$:

$$\begin{matrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{matrix} \quad \mapsto \quad \begin{matrix} q^{-1}(q-q^{-1})^2\alpha EF + \alpha K^{-1} & \alpha E \\ q^{-1}(q-q^{-1})^2\alpha KF & \alpha K\,. \end{matrix} \tag{42}$$

Such homomorphisms can be shown to exist by checking that the relations (36) hold inside $U_q(\mathfrak{sl}_2)$ for the desired images of the u_{ij}. For $n > 0$, consider the n-dimensional simple left $U_q(\mathfrak{sl}_2)$-module $V(n-1, +)$ defined in [8, I.4]. By using x and y as "raising" and "lowering" operators in the usual way, one can easily verify that the pullback $\overline{V}(n-1, +)$ of $V(n-1, +)$ along ψ_α is a simple $\mathcal{A}$-module. Identifying $\mathrm{ann}_{\overline{V}(n-1,+)}(u_{12})$ as "$\overline{m}_0$" from [8, I.4], which has a u-eigenvalue of αq^{n-1}, we conclude that $\overline{V}(n-1, +) \cong V_n(\alpha q^{n-1})$. This gives:

Theorem 91: *Assume that k is algebraically closed. Every finite dimensional simple left $\mathcal{A}$-module that is not annihilated by $u = u_{22}$ is the pullback of some simple left $U_q(\mathfrak{sl}_2)$-module along ψ_α for some $\alpha \in k^\times$.*

3.1.3 Finite Dimensional Weight Modules

Keep the notation and assumptions of the previous section. The *weight* $\mathcal{A}$-modules are the ones that decompose into simultaneous eigenspaces for the actions of u, t, and d; this is what it means to be semisimple over $R = k[u, t, d]$ when k is algebraically closed. In this section, we simply apply Theorem 45 to $\mathcal{A}$.

We observed in the previous section that the only maximal ideals $\mathfrak{m}(u_0, t_0, d_0)$ of R with finite σ-orbit are ones with $u_0 = 0$. Hence the *chain-type* finite-dimensional weight $\mathcal{A}$-modules are exactly the ones on which u acts as a unit.

In the previous section we identified the set $\mathscr{M}'_{\text{II}}$ of Definition 33 as

$$\begin{aligned} \mathscr{M}'_{\text{II}} &= \{\mathfrak{m} \in \max\operatorname{spec} R \mid \mathfrak{m} \text{ has infinite } \sigma\text{-orbit}, \ \sigma(z) \in \mathfrak{m}, \text{ and } \sigma^{-n+1}(z) \in \mathfrak{m} \text{ for some } n > 0\} \\ &= \{\mathfrak{m}(u_0,\ t_0 = (q^{-2n} + 1)u_0,\ d_0 = -q^{-2n}u_0^2) \mid u_0 \in k^\times \text{ and } n > 0\}. \end{aligned}$$

We will show that statement 4 of Theorem 45 holds for $\mathcal{A}$. Let $\mathfrak{m} = \mathfrak{m}(u_0, (q^{-2n} + 1)u_0, -q^{-2n}u_0^2)$ be an element of $\mathscr{M}'_{\text{II}}$. Suppose that $\sigma^{-n'+1}(z) \in \mathfrak{m}$, where $n' > 0$. Then, using (41), we have:

$$(q^{-2n} + 1)u_0 = (q^{-2n'} + 1)u'_0 \tag{43}$$

$$q^{-2n}u_0^2 = q^{-2n'}u_0'^2. \tag{44}$$

Using (43) to eliminate u'_0 from (44), we obtain

$$q^{-2n} = q^{-2n'} \left(\frac{q^{-2n} + 1}{q^{-2n'} + 1} \right)^2,$$

which simplifies to

$$(q^{2n} - q^{2n'}) = q^{-2n-2n'}(q^{2n} - q^{2n'}).$$

This requires that $n = n'$. Therefore Theorem 45 applies to $\mathcal{A}$ and gives:

Theorem 92: *Assume that k is algebraically closed. Finite-dimensional weight left $\mathcal{A}$-modules on which $u = u_{22}$ acts as a unit are semisimple.*

3.1.4 Prime Spectrum - Direct Approach

Much of this section documents the author's original approach to working out the prime spectrum of $\mathcal{A}$, which was done before the theory of section 2.6 became available. Section 3.1.5 below then provides a simpler, revised approach to the content here leading up Theorem 105. Many of the details here are still needed because they lead to a description of spec$(\mathcal{A})$ as a topological space and not just as a set.

We rely on the expression of $\mathcal{A}$ as a GWA in (39):

$$\begin{aligned} &k[u, t, d][x, y; \sigma, z] \\ &\sigma : u \mapsto q^2u, t \mapsto t, d \mapsto d \\ &z = d + q^{-2}tu - q^{-4}u^2. \end{aligned}$$

We can get at all the prime ideals of $\mathcal{A}$ by considering various quotients and localizations. Let us begin by laying out notation for the algebras to be considered:

- $\mathcal{A}/\langle u \rangle$ is simply a polynomial ring,

  $$\mathcal{A}/\langle u \rangle \cong k[u_{11}, u_{12}, u_{21}].$$

  A glance at the reflection equation relations (36) is enough to see this.

- Let $\mathcal{A}_u$ denote the localization of $\mathcal{A}$ at the set of powers of u, a denominator set because u is normal. By Proposition 22, this is $k[u^{\pm}, t, d][x, y; \sigma, z]$. By Proposition 12, $\mathcal{A}_u/\langle t, d\rangle = k[u^{\pm}][x, y; \sigma, z]$. In this quotient, z is a unit: $z = -q^{-4}u^2$. Hence, by Proposition 5,
$$\mathcal{A}_u/\langle t, d\rangle = k[u^{\pm}][x^{\pm}; \sigma].$$
- Let $\mathcal{A}_{ud}$ denote the localization of $\mathcal{A}_u$ at the set of powers of d, a denominator set because d is central. By Proposition 22, this is $k[u^{\pm}, t, d^{\pm}][x, y; \sigma, z]$. By Proposition 12,
$$\mathcal{A}_{ud}/\langle t\rangle = k[u^{\pm}, d^{\pm}][x, y; \sigma, z].$$
- Let $\mathcal{A}_{ut}$ denote the localization of $\mathcal{A}_u$ at the set of powers of t, a denominator set because t is central. By Proposition 22, this is $k[u^{\pm}, t^{\pm}, d][x, y; \sigma, z]$. Let $\mathcal{A}_{utx}$ denote the localization of the latter algebra at the set of powers of x; by Proposition 21, the set of powers of x is indeed a denominator set, and we obtain $\mathcal{A}_{utx} = k[u^{\pm}, t^{\pm}, d][x^{\pm}; \sigma]$. Then
$$\mathcal{A}_{utx}/\langle d\rangle = k[u^{\pm}, t^{\pm}][x^{\pm}; \sigma].$$
- Let $\mathcal{A}_{utxd}$ denote the localization of $\mathcal{A}_{utx}$ at the set of powers of d:
$$\mathcal{A}_{utxd} = k[u^{\pm}, t^{\pm}, d^{\pm}][x^{\pm}; \sigma].$$

What will turn out to be missing from this list is an algebra that gives us access to those prime ideals of $\mathcal{A}_{ut}$ that contain some power of x. We cover this in the next section.

3.1.4.1 Primes of $\mathcal{A}_{ut}$ That Contain a Power of x

We write $\mathcal{A}_{ut}$ as
$$\mathcal{A}_{ut} = R[x, y; \sigma, z],$$
where $R = k[u^{\pm}, t^{\pm}, d]$. A reminder about our notation: a subscript on a subset of a GWA indicates a certain subset of its base ring, seen in Definition 13. Define
$$r_n = (q^{2n} + 1)^2 d + q^{2n} t^2 \tag{45}$$
for $n \in \mathbb{Z}$; these elements of R will help us to understand the ideal of $\mathcal{A}_{ut}$ generated by a power of x:

Proposition 93: *Let $n \in \mathbb{Z}_{>0}$. Then*
$$\prod_{j=n-i+1}^{n} r_j \in \langle x^n \rangle_{n-i} \tag{46}$$
for all $0 \le i \le n$.

Proof: The induction will rely on the following observations:

1. For $n \ge 1$, $r_n \in \langle \sigma^n(z), z\rangle_R$.
2. Let I be an ideal of a GWA $R[x, y; \sigma, z]$. For $n \ge 1$, $(R^{\sigma} \cap I_n)\langle \sigma^n(z), z\rangle_R \subseteq I_{n-1}$.

Direct calculation verifies observation 1,
$$r_n = \frac{q^{2n+2}}{q^{2n} - 1} t u^{-1}(\sigma^n(z) - z) + \frac{q^{2n} + 1}{q^{2n} - 1}(q^{4n} z - \sigma^n(z)),$$
and observation 2 follows from Proposition 18. The $i = 0$ case, $1 \in \langle x^n \rangle_n$, is trivial. Assume that $0 \le i < n$ and that (46) holds for i. Then $a := r_n r_{n-1} \cdots r_{n-(i-1)} \in \langle x^n \rangle_{n-i}$. By observation 2, $a\langle \sigma^{n-i}(z), z\rangle_R \subseteq \langle x^n \rangle_{n-(i+1)}$. Hence, by observation 1, $a r_{n-i} \in \langle x^n \rangle_{n-(i+1)}$, proving (46) for $i + 1$. ■

Proposition 94: *Assume that $n \geq 1$ and $P \in \operatorname{spec}(\mathcal{A}_{ut})$. If $x^n \in P$ and $x^{n-1} \notin P$, then $r_n \in P$.*

Proof: From the $i = n$ case of Proposition 93,

$$r_1 r_2 \cdots r_n \in P.$$

Since this is a product of central elements in $\mathcal{A}_{ut}$, we conclude that that $r_{n'} \in P$ for some n'. In particular, $r_{n'} \in P_{n-1}$. Applying Proposition 93 with $i = 1$, we also have $r_n \in P_{n-1}$. Since t is a unit, and since q is not a root of unity, it is clear from (45) that $1 \in \langle r_n, r_{n'} \rangle_R$ if $n \neq n'$. We assumed that $x^{n-1} \notin P$, so $n' = n$. ■

So when considering homogeneous prime ideals P of $\mathcal{A}_{ut}$ that contain a power of x, we can eliminate a variable by factoring out the ideal generated by one of the r_i. Namely, we may factor out $\langle r_n \rangle$ if $n \geq 1$ is taken to be minimal such that $x^n \in P$, and we may then consider P as a prime ideal of $A_{(n)} := \mathcal{A}_{ut}/\langle r_n \rangle$. Using Proposition 12, this algebra is isomorphic to

$$k[u^{\pm}, t^{\pm}][x, y; \sigma, z_n],$$

where

$$z_n = \frac{-q^{2n}}{(q^{2n}+1)^2} t^2 + q^{-2} ut - q^{-4} u^2. \tag{47}$$

Let $R_{(n)}$ denote $k[u^{\pm}, t^{\pm}]$, thought of as $R/\langle r_n \rangle_R$. The ideal generated by x^n can be pinned down completely in $A_{(n)}$. We again start by defining some special elements of the base ring that will help us break things down. Make the following definitions:

$$\begin{aligned} & s_j^n = u - \frac{q^{2j}}{q^{2n}+1} t && \text{for } n, j \in \mathbb{Z}, \\ & \begin{cases} \mathcal{J}_m^n &= \{j \in \mathbb{Z} \mid 1 \leq j \leq n - m\} \\ \mathcal{J}_{-m}^n &= \{j \in \mathbb{Z} \mid m + 1 \leq j \leq n\} \end{cases} && \text{for } m \geq 0,\ n > 0, \\ & \pi_m^n = \textstyle\prod_{j \in \mathcal{J}_m^n} s_j^n && \text{for } m \in \mathbb{Z},\ n > 0. \end{aligned} \tag{48}$$

Here is a way to visually organize these definitions for the example $n = 3$:

$$\begin{array}{ccccccc} & & & 1 & & & = \pi_3^3 \\ & & & s_1^3 & & & = \pi_2^3 \\ & & s_1^3 & \cdot & s_2^3 & & = \pi_1^3 \\ s_1^3 & \cdot & s_2^3 & \cdot & s_3^3 & & = \pi_0^3 \\ & & s_2^3 & \cdot & s_3^3 & & = \pi_{-1}^3 \\ & & & s_3^3 & & & = \pi_{-2}^3 \\ & & & 1 & & & = \pi_{-3}^3 \end{array}$$

Observe that $\sigma(z_n) = -s_n^n s_0^n$ and that

$$\sigma^{-1}(s_j^n) = q^{-2} s_{j+1}^n, \tag{49}$$

so that

$$\sigma^i(z_n) = -q^{4i-4} s_{n-i+1}^n s_{1-i}^n \tag{50}$$

for $n, i \in \mathbb{Z}$. Finally, observe that the s_j^n are pairwise coprime over various j, since q is not a root of unity.

For the next results, we abstract this situation.

Proposition 95: *Let n be an integer with $n \geq 1$. Consider an arbitrary GWA $A = R[x, y; \sigma, z]$ over a commutative ring R. Assume that $(s_j)_{j\in\mathbb{Z}}$ is a sequence of elements of R with the following properties:*

1. *z is a unit multiple of $s_1 s_{n+1}$.*
2. *$\sigma^{-1}(s_j)$ is a unit multiple of s_{j+1}, for all $j \in \mathbb{Z}$.*
3. *$\langle s_i, s_j \rangle_R = R$ for all distinct $i, j \in \mathbb{Z}$.*

Define $\mathcal{J}_m$ as $\mathcal{J}_m^n$ is defined in (48) and let $\pi_m = \prod_{j\in\mathcal{J}_m} s_j$. Then we have

$$\langle x^n \rangle_m = \langle \pi_m \rangle_R$$

for $m \in \mathbb{Z}$.

Proof: The sequence of ideals $\langle \pi_m \rangle_R$ satisfies the conditions needed in Proposition 18 in order for $\bigoplus_{m\in\mathbb{Z}} \langle \pi_m \rangle_R v_m$ to define an ideal of A, as can be checked using our assumptions 1 and 2. Since $\pi_n = 1$, the latter ideal contains x^n. This gives the inclusion $\langle x^n \rangle_m \subseteq \langle \pi_m \rangle_R$ for $m \in \mathbb{Z}$. To get equality we must show that

$$\pi_m \in \langle x^n \rangle_m \tag{51}$$

for all $m \in \mathbb{Z}$.

For $m \geq n$, (51) holds trivially. Assume that (51) holds for a given m, with $1 \leq m \leq n$. Then:

$$\begin{aligned}
\langle x^n \rangle_{m-1} &\supseteq \left\langle \pi_m \sigma^m(z), \sigma^{-1}(\pi_m) z \right\rangle_R &(52)\\
&= \left\langle \left(\prod_{j=1}^{n-m} s_j\right)(s_{n-(m-1)} s_{1-m}), \left(\prod_{j=2}^{n-(m-1)} s_j\right)(s_{n+1} s_1) \right\rangle_R &(53)\\
&= \pi_{m-1} \langle s_{1-m}, s_{n+1} \rangle_R = \pi_{m-1} R. &(54)
\end{aligned}$$

Line (52) is due to the induction hypothesis and Proposition 18. Line (53) uses assumptions 1 and 2. And line (54) uses assumption 3. Hence, by induction, (51) holds for $m \geq 0$.

Now assume that $1 - n \leq m \leq 0$ and that (51) holds for m. We can apply a similar strategy to what was done for (52)-(54):

$$\begin{aligned}
\langle x^n \rangle_{m-1} \supseteq \left\langle \pi_m, \sigma^{-1}(\pi_m) \right\rangle_R &= \left\langle \prod_{j=-m+1}^{n} s_j \ , \ \prod_{j=-m+2}^{n+1} s_j \right\rangle_R \\
&= \pi_{m-1} \langle s_{-m+1}, s_{n+1} \rangle_R = \pi_{m-1} R.
\end{aligned}$$

Hence, by induction, (51) holds for all $m \geq -n$. In particular (the case $m = -n$), $y^n \in \langle x^n \rangle$. Thus (51) holds trivially for $m < -n$. ∎

Corollary 96: *In the setup of Proposition 95, $\langle x^n \rangle = \langle y^n \rangle$.*

Proof: We shall make use of Proposition 3 to exploit symmetries in the hypotheses of Proposition 95. Let us use hats to denote our new batch of input data to Proposition 95. Consider A as a GWA $R[\hat{x}, \hat{y}; \hat{\sigma}, \hat{z}]$, with elements $(\hat{s}_j)_{j\in\mathbb{Z}}$ of R, where

$$\begin{array}{lll}
\hat{x} = y & \hat{y} = x & \hat{z} = \sigma(z) \\
\hat{\sigma} = \sigma^{-1} & \hat{s}_j = s_{n+1-j} &
\end{array} \tag{55}$$

This satisfies the hypotheses of Proposition 95. Following along the notations needed to state the conclusion, define

$$\hat{\pi}_m = \prod_{j \in \mathcal{J}_m} \hat{s}_j$$

and also define

$$\hat{I}_m = I_{-m} \qquad \text{(which is } \{r \in R \mid rv_{-m} \in I\}\text{)} \tag{56}$$

whenever $I \subseteq A$, to match Definition 13 with the new GWA structure. Observe that

$$\{n+1-j \mid j \in \mathcal{J}_m\} = \mathcal{J}_{-m} \tag{57}$$

for all $m \in \mathbb{Z}$, so that $\hat{\pi}_m = \pi_{-m}$. The conclusion of Proposition 95 is then that $\widehat{\langle \hat{x}^n \rangle}_m = \langle \hat{\pi}_m \rangle_R$. That is,

$$\langle y^n \rangle_{-m} = \langle \pi_{-m} \rangle_R$$

for all $m \in \mathbb{Z}$. ■

In order to get at the homogeneous primes of $A_{(n)}$ that contain x^n, we now seek to describe *all* the homogeneous ideals of $A_{(n)}$ that contain x^n. Statements of the next few results remain in a general GWA setting, in order to continue taking advantage of the symmetry of GWA expressions.

Proposition 97: *Assume the setup of Proposition 95. Fix arbitrary integers $\ell_1 \leq \ell_2$. There is an element e_0 of R such that, setting $e_j = \sigma^{-j}(e_0)$ for $j \in \mathbb{Z}$, the family $(e_j)_{j \in \mathbb{Z}}$ satisfies:*

1. *$e_j \equiv 1$ mod s_j for $j \in \mathbb{Z}$.*
2. *$e_j \equiv 0$ mod s_i for distinct $i, j \in \{\ell_1, \ldots, \ell_2\}$.*
3. *$\bar{e}_{\ell_1}, \ldots, \bar{e}_{\ell_2}$ is a collection of orthogonal idempotents that sum to 1, where bars denote cosets with respect to $\left\langle \prod_{i=\ell_1}^{\ell_2} s_i \right\rangle$.*

Proof: The s_j, for $j \in \mathbb{Z}$, are pairwise coprime as elements of R. The Chinese Remainder Theorem (CRT) provides an $e_0 \in R$ which is congruent to 1 mod s_0 and congruent to 0 mod s_i for all nonzero $i \in \{\ell_1 - \ell_2, \ldots, \ell_2 - \ell_1\}$. Then for $j \in \mathbb{Z}$ we have that $\sigma^{-j}(e_0)$ is congruent to 1 mod s_j and congruent to 0 mod s_i for all $i \in \{\ell_1 - \ell_2 + j, \ldots, \ell_2 - \ell_1 + j\}$ with $i \neq j$. Setting $e_j = \sigma^{-j}(e_0)$ gives us 1 and 2. Part of the CRT says that

$$\left\langle \prod_{i=\ell_1}^{\ell_2} s_i \right\rangle = \bigcap_{i=\ell_1}^{\ell_2} \langle s_i \rangle,$$

and 3 easily follows from this using 1 and 2. ■

Proposition 98: *Assume the setup of Proposition 95. Let $(e_j)_{j \in \mathbb{Z}}$ be as in Proposition 97 with $\ell_1 = 1$ and $\ell_2 = n$. There are mutually inverse inclusion-preserving bijections*

$$\begin{array}{ccc}
\left\{ \begin{array}{l} \textit{homogeneous right } R[x;\sigma]\textit{-submodules } I \\ \textit{of } A \textit{ containing } x^n \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \textit{families } (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \textit{ of ideals} \\ \textit{of } R \textit{ satisfying (59) with } s_j \in I_{mj} \textit{ for all} \\ m, j \end{array} \right\} \\
I & \mapsto & (I_m + \langle s_j \rangle \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \\
\bigoplus_{m \in \mathbb{Z}} \left(\langle \pi_m \rangle + \sum_{j \in \mathcal{J}_m} I_{mj} e_j \right) v_m & \leftarrow & (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m),
\end{array} \tag{58}$$

where the condition (59) is that

$$I_{-(m+1),j} \subseteq I_{-m,j} \ \forall j \in \mathcal{J}_{-(m+1)} \qquad \textit{and} \qquad I_{mj} \subseteq I_{m+1,j} \ \forall j \in \mathcal{J}_{m+1} \tag{59}$$

for all $m \in \mathbb{Z}_{\geq 0}$.

Proof: Combining Propositions 15 and 95, we obtain the following correspondence:

$$\begin{array}{ccc}
\left\{\begin{array}{l}\text{homogeneous right } R[x;\sigma]\text{-submodules} \\ I \text{ of } A \text{ containing } x^n\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{l}\text{sequences } (I_m)_{m\in\mathbb{Z}} \text{ of ideals of } R \text{ satis-} \\ \text{fying the conditions (5) of Proposition} \\ \text{15 with } \pi_m \in I_m \text{ for all } m\end{array}\right\} \\
I & \mapsto & (I_m)_{m\in\mathbb{Z}} \\
\bigoplus_{m\in\mathbb{Z}} I_m v_m & \leftarrow\!\shortmid & (I_m)_{m\in\mathbb{Z}}.
\end{array} \tag{60}$$

The s_j, for $j \in \mathbb{Z}$, are pairwise coprime as elements of R. So an ideal I_m of R containing π_m corresponds, via the CRT, to a collection of ideals $(I_{mj})_{j\in\mathcal{J}_m}$ such that $s_j \in I_{mj}$ for $j \in \mathcal{J}_m$. Using Proposition 144, the correspondence is:

$$\begin{array}{ccc}
\left\{\begin{array}{l}\text{sequences } (I_m)_{m\in\mathbb{Z}} \text{ of ideals of } R \text{ with} \\ \pi_m \in I_m \text{ for all } m\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{l}\text{families } (I_{mj} \mid m\in\mathbb{Z},\ j\in\mathcal{J}_m) \text{ of ide-} \\ \text{als of } R \text{ with } s_j \in I_{mj} \text{ for all } m, j\end{array}\right\} \\
(I_m)_{m\in\mathbb{Z}} & \mapsto & (I_{mj} = I_m + \langle s_j\rangle \mid m\in\mathbb{Z},\ j\in\mathcal{J}_m) \\
\left(\langle\pi_m\rangle + \sum_{j\in\mathcal{J}_m} I_{mj}e_j\right)_{m\in\mathbb{Z}} & \leftarrow\!\shortmid & (I_{mj} \mid m\in\mathbb{Z},\ j\in\mathcal{J}_m).
\end{array} \tag{61}$$

In order to make use of this with (60), we need to express the condition (5) of Proposition 15 in terms of the I_{mj}. Let $(I_m)_{m\in\mathbb{Z}}$ be a sequence of ideals of R with $\pi_m \in I_m$ for all m, and let $(I_{mj} \mid m\in\mathbb{Z},\ j\in\mathcal{J}_m)$ be the family of ideals it corresponds to in (61). For $m \in \mathbb{Z}_{\geq 0}$,

$$I_m \subseteq I_{m+1} \Leftrightarrow \langle\pi_m\rangle + \sum_{j=1}^{n-m} I_{mj}e_j \subseteq \langle\pi_{m+1}\rangle + \sum_{j=1}^{n-m-1} I_{m+1,j}e_j \tag{62}$$

$$\Rightarrow \langle s_i\rangle + I_{mi} \subseteq \langle s_i\rangle + I_{m+1,i} \qquad \forall i \in \mathcal{J}_{m+1} \tag{63}$$

$$\Rightarrow I_{mi} \subseteq I_{m+1,i} \qquad \forall i \in \mathcal{J}_{m+1} \tag{64}$$

$$\Rightarrow I_m \subseteq I_{m+1}. \tag{65}$$

Line (63) is obtained by adding $\langle s_i\rangle$ to both sides of the inclusion in line (62), and using the properties of the e_j from Proposition 97. Line (64) is due to the fact that $s_i \in I_{m+1,i}$. Line (65) can be seen by looking at (62) and noting that $e_{n-m} \in \langle\pi_{m+1}\rangle$ because e_{n-m} vanishes mod s_j for $j \in \mathcal{J}_{m+1}$. For similar reasons we also have, for $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
&I_{-(m+1)}\sigma^{-m}(z) \subseteq I_{-m} \\
&\quad\Leftrightarrow \left(\langle\pi_{-(m+1)}\rangle + \sum_{j=m+2}^{n} I_{-(m+1),j}e_j\right) s_{n+m+1}s_{m+1} \subseteq \langle\pi_{-m}\rangle + \sum_{j=m+1}^{n} I_{-m,j}e_j \\
&\quad\Leftrightarrow s_{n+m+1}\langle\pi_{-m}\rangle + \sum_{j=m+2}^{n} I_{-(m+1),j}s_{n+m+1}s_{m+1}e_j \subseteq \langle\pi_{-m}\rangle + \sum_{j=m+1}^{n} I_{-m,j}e_j \\
&\quad\Rightarrow \langle s_i\rangle + I_{-(m+1),i}s_{n+m+1}s_{m+1} \subseteq \langle s_i\rangle + I_{-m,i} \qquad \forall i \in \mathcal{J}_{-(m+1)} \\
&\quad\Rightarrow I_{-(m+1),i} \subseteq I_{-m,i} \qquad \forall i \in \mathcal{J}_{-(m+1)} \qquad (66) \\
&\quad\Rightarrow I_{-(m+1)}\sigma^{-m}(z) \subseteq I_{-m}.
\end{aligned}$$

The only subtlety this time is that line (66) relies on the fact that s_{n+m+1} and s_{m+1} are units modulo s_i for all $i \in \mathcal{J}_{-(m+1)}$.

We conclude that the condition (5) of Proposition 15 holds for $(I_m)_{m\in\mathbb{Z}}$ if and only if the condition (59) holds for $(I_{mj} \mid m\in\mathbb{Z},\ j\in\mathcal{J}_m)$. Combining this fact with the correspondences (60) and (61) yields the desired correspondence (58). ■

Corollary 99: *Assume the setup of Propositions 95 and 98. There are mutually inverse inclusion-preserving bijections*

$$\begin{array}{ccc} \left\{ \begin{array}{l} \textit{homogeneous right } R[y;\sigma^{-1}]\textit{-submodules } I \\ \textit{of } A \textit{ containing } y^n \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \textit{families } (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \textit{ of ideals} \\ \textit{of } R \textit{ satisfying (68) with } s_j \in I_{mj} \textit{ for all} \\ m,j \end{array} \right\} \\ I & \mapsto & (I_m + \langle s_j\rangle \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \\ \bigoplus_{m\in\mathbb{Z}} \left(\langle \pi_m\rangle + \sum_{j\in\mathcal{J}_m} I_{mj}e_j\right) v_m & \leftarrow\!\shortmid & (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m), \end{array} \tag{67}$$

where the condition (68) is that

$$I_{-(m+1),j} \supseteq I_{-m,j}\ \forall j \in \mathcal{J}_{-(m+1)} \qquad \textit{and} \qquad I_{mj} \supseteq I_{m+1,j}\ \forall j \in \mathcal{J}_{m+1} \tag{68}$$

for all $m \in \mathbb{Z}_{\geq 0}$.

Proof: We shall apply Proposition 98 while viewing A as a GWA with the alternative GWA structure $R[y, x; \sigma^{-1}, \sigma(z)]$. Make the definitions (55)-(56), and also define

$$\hat{e}_j = e_{n+1-j}.$$

These data satisfy the hypotheses of Proposition 98, and allows us to conclude that there is a correspondence

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{homogeneous right } R[y;\sigma^{-1}]\text{-submodules} \\ I \text{ of } A \text{ containing } y^n \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \text{families } (\hat{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \text{ of ide-} \\ \text{als of } R \text{ satisfying (69) with } \hat{s}_j \in \hat{I}_{mj} \\ \text{for all } m,j \end{array} \right\} \\ I & \mapsto & (\hat{I}_m + \langle \hat{s}_j\rangle \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \\ \bigoplus_{m\in\mathbb{Z}} \left(\langle \hat{\pi}_m\rangle + \sum_{j\in\mathcal{J}_m} \hat{I}_{mj}\hat{e}_j\right) v_m & \leftarrow\!\shortmid & (\hat{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m), \end{array}$$

where the condition (69) is that

$$\hat{I}_{-(m+1),j} \subseteq \hat{I}_{-m,j}\ \forall j \in \mathcal{J}_{-(m+1)} \qquad \text{and} \qquad \hat{I}_{mj} \subseteq \hat{I}_{m+1,j}\ \forall j \in \mathcal{J}_{m+1} \tag{69}$$

for all $m \in \mathbb{Z}_{\geq 0}$. Using the observation (57) and reindexing by $(m, j) \mapsto (-m, n+1-j)$, this becomes the correspondence (67). ■

The following Proposition is not strictly needed to eventually see that all ideals of $\mathcal{A}_u$ are homogeneous– it is sufficient to observe that conjugation by u in a linear map $\mathcal{A}_u \to \mathcal{A}_u$ that has the graded components of $\mathcal{A}_u$ as distinct eigenspaces (see Proposition 81). However, it is interesting to note that homogeneity is guaranteed even in the abstract setup of Proposition 95.

Proposition 100: *Assume the setup of Proposition 95. All ideals of A containing x^n are homogeneous.*

Proof: Let $(e_j)_{j\in\mathbb{Z}}$ be as in Proposition 97 with $\ell_1 = -n+2$ and $\ell_2 = 2n-1$. Let I be any ideal of A containing x^n, and let $\sum_{m\in\mathbb{Z}} a_m v_m \in I$ be an arbitrary element. Then since $\langle x^n\rangle = \langle y^n\rangle$, from Corollary 96, we have $v_m \in I$ for $m \geq n$ and for $m \leq -n$, and the problem is reduced to considering

$$\sum_{m=-n+1}^{n-1} a_m v_m \in I$$

and needing to show that $a_m v_m \in I$ for $m \in \{-n+1, \ldots, n-1\}$. Consider any $j, j' \in \{1, \ldots, n\}$. Multiplying on the left by e_j and on the right by $e_{j'}$ yields

$$I \ni \sum_{m=-n+1}^{n-1} e_j a_m v_m e_{j'} = \sum_{m=-n+1}^{n-1} a_m e_j \sigma^m(e_{j'}) v_m = \sum_{m=-n+1}^{n-1} a_m e_j e_{j'-m} v_m.$$

When $-n+1 \le m \le n-1$ and $1 \le j' \le n$, we have $-n+2 \le j'-m \le 2n-1$. So, since $\pi_0 \in I$ (due to Proposition 95), the product $e_j e_{j'-m}$ that appears above vanishes mod I unless $j'-m=j$, in which case it is congruent to e_j mod I. Thus we have

$$a_{j'-j} e_j v_{j'-j} \in I$$

for all $j, j' \in \{1, \ldots, n\}$. When $j \in \mathcal{J}_m$, we have $j, m+j \in \{1, \ldots, n\}$, so this shows that $a_m e_j v_m \in I$, for all $m \in \{-n+1, \ldots, n-1\}$ and $j \in \mathcal{J}_m$, and in particular that

$$\sum_{j \in \mathcal{J}_m} a_m e_j v_m \in I.$$

Fix an $m \in \{-n+1, \ldots, n-1\}$. Since $\pi_m \in I_m$ (due to Proposition 95), $\sum_{j \in \mathcal{J}_m} e_j \equiv 1 \bmod I_m$. Hence $a_m v_m \in I$. ■

Corollary 101: *Assume the setup of Propositions 95 and 98. There are mutually inverse inclusion-preserving bijections*

$$\begin{array}{ccc} \{ \text{ ideals } I \text{ of } A \text{ containing } x^n \ \} & \leftrightarrow & \left\{ \begin{array}{l} \text{families } (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \text{ of ideals} \\ \text{of } R \text{ satisfying (71) with } s_j \in I_{mj} \text{ for all} \\ m, j \end{array} \right\} \\ I & \mapsto & (I_m + \langle s_j \rangle \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \\ \bigoplus_{m \in \mathbb{Z}} \left(\langle \pi_m \rangle + \sum_{j \in \mathcal{J}_m} I_{mj} e_j \right) v_m & \leftarrow & (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m), \end{array} \tag{70}$$

where the condition (71) is that

$$\begin{array}{lcl} I_{-(m+1),j} = I_{-m,j}\ \forall j \in \mathcal{J}_{-(m+1)}, & & I_{mj} = I_{m+1,j}\ \forall j \in \mathcal{J}_{m+1}, \\ \sigma(I_{-(m+1),j}) = I_{-m,j-1}\ \forall j \in \mathcal{J}_{-(m+1)}, & \text{and} & \sigma(I_{m,j+1}) = I_{m+1,j}\ \forall j \in \mathcal{J}_{m+1} \end{array} \tag{71}$$

for all $m \in \mathbb{Z}_{\ge 0}$.

Proof: We shall deduce left-handed versions of (58) and (67) by viewing A^{op} as a GWA $R[x, y; \sigma^{-1}, \sigma(z)]$. Recall the notation I_m^{op} from Definition 13 and Remark 14. Defining

$$\begin{array}{llll} \hat{x} = x & \hat{y} = y & \hat{z} = \sigma(z) & \\ \hat{\sigma} = \sigma^{-1} & \hat{s}_j = s_{n+1-j} & \hat{e}_j = e_{n+1-j} & \hat{\pi}_m = \prod_{j \in \mathcal{J}_m} \hat{s}_j, \end{array}$$

we may now write $A^{\mathrm{op}} = R[\hat{x}, \hat{y}; \hat{\sigma}, \hat{z}]$. These data satisfy the hypotheses of Proposition 98 and Corollary 99, so we obtain correspondences

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{homogeneous left } R[x;\sigma]\text{-submodules} \\ \text{of } A \text{ containing } x^n \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \text{families } (\hat{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \text{ of ide-} \\ \text{als of } R \text{ satisfying (73) with } \hat{s}_j \in \hat{I}_{mj} \\ \text{for all } m, j \end{array} \right\} \\ \left\{ \begin{array}{l} \text{homogeneous left } R[y;\sigma^{-1}]\text{-submodules} \\ \text{of } A \text{ containing } y^n \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \text{families } (\hat{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \text{ of ide-} \\ \text{als of } R \text{ satisfying (74) with } \hat{s}_j \in \hat{I}_{mj} \\ \text{for all } m, j \end{array} \right\} \\ I & \mapsto & (I_m^{\mathrm{op}} + \langle \hat{s}_j \rangle \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \\ \bigoplus_{m \in \mathbb{Z}} \sigma^m \left(\langle \hat{\pi}_m \rangle + \sum_{j \in \mathcal{J}_m} \hat{I}_{mj} \hat{e}_j \right) v_m & \leftarrow & (\hat{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m), \end{array} \tag{72}$$

where the specified conditions are that

$$\hat{I}_{-(m+1),j} \subseteq \hat{I}_{-m,j}\ \forall j \in \mathcal{J}_{-(m+1)}, \qquad \hat{I}_{mj} \subseteq \hat{I}_{m+1,j}\ \forall j \in \mathcal{J}_{m+1}, \tag{73}$$

$$\hat{I}_{-(m+1),j} \supseteq \hat{I}_{-m,j} \ \forall j \in \mathcal{J}_{-(m+1)}, \qquad \text{and} \qquad \hat{I}_{mj} \supseteq \hat{I}_{m+1,j} \ \forall j \in \mathcal{J}_{m+1} \tag{74}$$

for all $m \in \mathbb{Z}_{\geq 0}$. To make this useful, we transform the expression for the families of ideals in the right hand side of (72) as follows:

$$I_{mj} = \sigma^m(\hat{I}_{m,n+1-(j+m)}).$$

The index sets $\mathcal{J}_m$ have symmetries that can be used to reindex sums and products after applying this transformation: $\mathcal{J}_m = \{n+1-j \mid j \in \mathcal{J}_{-m}\} = \{j-m \mid j \in \mathcal{J}_{-m}\}$. A consequence is that $\sigma^m(\hat{\pi}_m)$ is a unit multiple of π_m. One now makes the routine substitutions and reindexings in (72)-(74) to obtain correspondences

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{homogeneous left } R[x;\sigma]\text{-submodules} \\ \text{of } A \text{ containing } x^n \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \text{families } (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \text{ of ide-} \\ \text{als of } R \text{ satisfying (76) with } s_j \in I_{mj} \\ \text{for all } m, j \end{array} \right\} \\ \left\{ \begin{array}{l} \text{homogeneous left } R[y;\sigma^{-1}]\text{-submodules} \\ \text{of } A \text{ containing } y^n \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \text{families } (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \text{ of ide-} \\ \text{als of } R \text{ satisfying (77) with } s_j \in I_{mj} \\ \text{for all } m, j \end{array} \right\} \\ I & \mapsto & (I_m + \langle s_j \rangle \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m) \\ \bigoplus_{m\in\mathbb{Z}} \left(\langle \pi_m \rangle + \sum_{j\in\mathcal{J}_m} I_{mj} e_j \right) v_m & \leftarrow & (I_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}_m), \end{array} \tag{75}$$

where the specified conditions are now that

$$\sigma(I_{-(m+1),j}) \subseteq I_{-m,j-1} \ \forall j \in \mathcal{J}_{-(m+1)}, \qquad \sigma(I_{m,j+1}) \subseteq I_{m+1,j} \ \forall j \in \mathcal{J}_{m+1}, \tag{76}$$

$$\sigma(I_{-(m+1),j}) \supseteq \hat{I}_{-m,j-1} \ \forall j \in \mathcal{J}_{-(m+1)}, \qquad \text{and} \qquad \sigma(I_{m,j+1}) \supseteq I_{m+1,j} \ \forall j \in \mathcal{J}_{m+1} \tag{77}$$

for all $m \in \mathbb{Z}_{\geq 0}$.

Note that, by Corollary 96, an ideal of A contains x^n if and only if it contains y^n. And note that, by Proposition 100, all ideals of A are homogeneous. Hence we may combine (58), (67), and (75) to obtain the correspondence (70), and the condition in (71) is just the conjunction of conditions (59), (68), (76), and (77). ■

We now specialize back to the algebra $A_{(n)} = \mathcal{A}_{ut}/\langle r_n \rangle$. Corollary 101 applies to $A_{(n)}$ with the elements of $R_{(n)}$ defined in (48).

Proposition 102: *For $n \geq 1$, there are mutually inverse inclusion-preserving bijections*

$$\begin{array}{ccc} \{ \text{ ideals } I/\langle x^n \rangle \text{ of } A_{(n)}/\langle x^n \rangle \} & \leftrightarrow & \{ \text{ ideals } \widetilde{I}_{\cdot\cdot} \text{ of } k[t^{\pm}] \} \\ I/\langle x^n \rangle & \mapsto & (I_0 + \langle s_1^n \rangle_{R_{(n)}}) \cap k[t^{\pm}] \\ \left(\bigoplus_{m\in\mathbb{Z}} \left(\langle \pi_m^n \rangle + \left\langle \widetilde{I}_{\cdot\cdot} \right\rangle \right) v_m \right) / \langle x^n \rangle & \leftarrow & \widetilde{I}_{\cdot\cdot}\,. \end{array} \tag{78}$$

Proof: Let e_j^n for $j \in \mathbb{Z}$ be as in Proposition 97 with $\ell_1 = 1$ and $\ell_2 = n$. In particular they are elements of $R_{(n)}$ such that e_j^n is congruent to 1 mod s_j^n for $j \in \mathbb{Z}$ and congruent to 0 mod s_i^n for all distinct $i, j \in \{1, \ldots, n\}$, and $\sigma^{-1}(e_j^n) = e_{j+1}^n$ for all $j \in \mathbb{Z}$. For $j \in \mathbb{Z}$, the algebra $R_{(n)}/\langle s_j^n \rangle$ is isomorphic to $k[t^{\pm}]$, and the isomorphism is the composite

$$k[t^{\pm}] \hookrightarrow R_{(n)} \twoheadrightarrow R_{(n)}/\langle s_j^n \rangle.$$

We obtain from this a correspondence of ideals for each j:

$$\begin{array}{ccc} \{\text{ideals of } R_{(n)} \text{ containing } s_j^n\} & \leftrightarrow & \{\text{ideals of } k[t^{\pm}]\} \\ J & \mapsto & \widetilde{J} = J \cap k[t^{\pm}] \\ \widetilde{J} + \langle s_j^n \rangle & \leftarrow & \widetilde{J}. \end{array} \tag{79}$$

This allows us to restate the correspondence that we obtain from Corollary 101 as

$$\begin{array}{ccc} \{\text{ ideals } I \text{ of } A_{(n)} \text{ containing } x^n \ \} & \leftrightarrow & \left\{ \begin{array}{l} \text{families } (\widetilde{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}^n_m) \text{ of ide-} \\ \text{als of } k[t^{\pm}] \text{ satisfying (81)} \end{array} \right\} \\ I & \mapsto & ((I_m + \langle s^n_j \rangle) \cap k[t^{\pm}] \mid m \in \mathbb{Z},\ j \in \mathcal{J}^n_m) \\ \bigoplus_{m\in\mathbb{Z}} \left(\langle \pi^n_m \rangle + \sum_{j \in \mathcal{J}^n_m} (\widetilde{I}_{mj} + \langle s^n_j \rangle) e^n_j \right) v_m & \leftarrow & (\widetilde{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}^n_m), \end{array} \tag{80}$$

where the condition (81) is that

$$\begin{array}{lcl} \widetilde{I}_{-(m+1),j} = \widetilde{I}_{-m,j} \ \forall j \in \mathcal{J}^n_{-(m+1)}, & & \widetilde{I}_{mj} = \widetilde{I}_{m+1,j} \ \forall j \in \mathcal{J}^n_{m+1}, \\ \widetilde{I}_{-(m+1),j} = \widetilde{I}_{-m,j-1} \ \forall j \in \mathcal{J}^n_{-(m+1)}, & \text{and} & \widetilde{I}_{m,j+1} = \widetilde{I}_{m+1,j} \ \forall j \in \mathcal{J}^n_{m+1} \end{array} \tag{81}$$

for all $m \in \mathbb{Z}_{\geq 0}$. The σ has disappeared from the condition (71) because σ fixes $k[t^{\pm}]$. Notice that (81) simply says that all the ideals in the family $(\widetilde{I}_{mj} \mid m \in \mathbb{Z},\ j \in \mathcal{J}^n_m)$ are equal. So we may as well give them all one name, $\widetilde{I}_{\cdot\cdot} := \widetilde{I}_{01}$. We may also simplify the expression of the left hand side of (80): for $m \in \mathbb{Z}$, we have

$$\begin{aligned} \langle \pi^n_m \rangle + \sum_{j \in \mathcal{J}^n_m} (\widetilde{I}_{\cdot\cdot} + \langle s^n_j \rangle) e^n_j &= \langle \pi^n_m \rangle + \langle s^n_j e^n_j \mid j \in \mathcal{J}^n_m \rangle + \sum_{j \in \mathcal{J}^n_m} \left\langle \widetilde{I}_{\cdot\cdot} \right\rangle e^n_j \\ &= \langle \pi^n_m \rangle + \langle s^n_j e^n_j \mid j \in \mathcal{J}^n_m \rangle + \left\langle \widetilde{I}_{\cdot\cdot} \right\rangle && (82) \\ &= \langle \pi^n_m \rangle + \left\langle \widetilde{I}_{\cdot\cdot} \right\rangle. && (83) \end{aligned}$$

Line (82) is due to the fact that $\sum_{j \in \mathcal{J}^n_m} e^n_j$ is congruent to 1 mod π^n_m. Line (83) is due to the fact that $s^n_j e^n_j$ is congruent to 0 mod π^n_m for all $j \in \mathcal{J}^n_m$. Thus we obtain (78). ■

Proposition 103: *Products of ideals are preserved by the correspondence (78).*

Proof: Let $\mathfrak{a}, \mathfrak{b}$ be ideals of $k[t^{\pm}]$, and let $I/\langle x^n \rangle, J/\langle x^n \rangle$ be the respective corresponding ideals of $A_{(n)}/\langle x^n \rangle$ via (78). We must show that the product $\mathfrak{ab}$ corresponds via (78) to $(I/\langle x^n \rangle)(J/\langle x^n \rangle) = (IJ + \langle x^n \rangle)/\langle x^n \rangle$. That is, we must show that $((IJ + \langle x^n \rangle)_0 + \langle s^n_1 \rangle) \cap k[t^{\pm}] = \mathfrak{ab}$. Using the fact that all of the ideals on the right hand side of (80) are equal,

$$\begin{array}{l} (I_m + \langle s^n_j \rangle) \cap k[t^{\pm}] = \mathfrak{a} \\ (J_m + \langle s^n_j \rangle) \cap k[t^{\pm}] = \mathfrak{b} \end{array} \tag{84}$$

for all $m \in \mathbb{Z},\ j \in \mathcal{J}^n_m$. The contraction $(IJ)_0$ of the product IJ consists of sums of products of homogeneous terms of opposite degree; i.e. terms of the form

$$(av_m) \cdot (bv_{-m}) = a\sigma^m(b)[\![m, -m]\!]$$

for $m \in \mathbb{Z}$. Hence $(IJ + \langle x^n \rangle)_0 + \langle s^n_1 \rangle$ can be written as

$$(IJ + \langle x^n \rangle)_0 + \langle s^n_1 \rangle = (IJ)_0 + \langle \pi^n_0 \rangle + \langle s^n_1 \rangle = (IJ)_0 + \langle s^n_1 \rangle = \sum_{m \in \mathbb{Z}} [\![m, -m]\!] I_m \sigma^m(J_{-m}) + \langle s^n_1 \rangle.$$

Observe the following:

> **Claim:** If $m \in \{0, \ldots, n-1\}$, then $[\![m, -m]\!]$ is a unit mod $\langle s^n_1 \rangle_{R_{(n)}}$. Otherwise, it is in $\langle s^n_1 \rangle_{R_{(n)}}$.
>
> **Proof:** If $m < 0$, then $[\![m, -m]\!] = \sigma^{[m+1,0]}(z_n)$ is divisible by z_n, which is divisible by s^n_1. If $m > n-1$, then $[\![m, -m]\!] = \sigma^{[1,m]}(z_n)$ is divisible by $\sigma^n(z_n)$, which is also divisible by s^n_1. If $m = 0$, then $[\![m, -m]\!] = 1$ is a unit mod s^n_1. Finally, assume that $m \in \{1, \ldots, n-1\}$. Then $[\![m, -m]\!] = \sigma^{[1,m]}(z_n)$ is a unit multiple of the product
>
> $$\prod_{i=1}^m s^n_{n-i+1} \prod_{i=1}^m s^n_{1-i}.$$

Observe that the assumption $1 \leq m \leq n-1$ precludes s_1^n from being a factor in the product above. Since the s_j^n are pairwise coprime, it follows that $[\![m, -m]\!]$ is a unit mod s_1^n.

This simplifies the expression above:

$$\sum_{m \in \mathbb{Z}} [\![m, -m]\!] I_m \sigma^m(J_{-m}) + \langle s_1^n \rangle = \sum_{m=0}^{n-1} I_m \sigma^m(J_{-m}) + \langle s_1^n \rangle.$$

Now we calculate what is needed:

$$\begin{aligned}
((IJ + \langle x^n \rangle)_0 + \langle s_1^n \rangle) \cap k[t^{\pm}] &= \left(\sum_{m=0}^{n-1} I_m \sigma^m(J_{-m}) + \langle s_1^n \rangle \right) \cap k[t^{\pm}] \\
&= \left(\sum_{m=0}^{n-1} (I_m + \langle s_1^n \rangle) \sigma^m(J_{-m} + \langle s_{m+1}^n \rangle) + \langle s_1^n \rangle \right) \cap k[t^{\pm}] \\
&= \left(\sum_{m=0}^{n-1} (\mathfrak{a} + \langle s_1^n \rangle) \sigma^m(\mathfrak{b} + \langle s_{m+1}^n \rangle) + \langle s_1^n \rangle \right) \cap k[t^{\pm}] \qquad (85) \\
&= \left(\sum_{m=0}^{n-1} \mathfrak{a}\mathfrak{b} + \langle s_1^n \rangle \right) \cap k[t^{\pm}] \\
&= (\mathfrak{a}\mathfrak{b} + \langle s_1^n \rangle) \cap k[t^{\pm}] \\
&= \mathfrak{a}\mathfrak{b}. \qquad (86)
\end{aligned}$$

Line (85) uses (84), and lines (85) and (86) both make use of the correspondence (79). ■

Corollary 104: *For $n \geq 1$, there is a homeomorphism*

$$\operatorname{spec}(A_{(n)}/\langle x^n \rangle) \;\approx\; \operatorname{spec}(k[t^{\pm}])$$

given by

$$\begin{array}{ccc}
P/\langle x^n \rangle & \mapsto & (P_0 + \langle s_1^n \rangle_{R_{(n)}}) \cap k[t^{\pm}] \\
\bigoplus_{m \in \mathbb{Z}} (\langle \pi_m^n \rangle + \langle \mathfrak{p} \rangle)\, v_m \;/\; \langle x^n \rangle & \leftarrow\!\shortmid & \mathfrak{p}\;.
\end{array} \qquad (87)$$

Proof: This follows from Propositions 102 and 103. ■

3.1.4.2 The Prime Spectrum of $\mathcal{A}$

Express the algebra $\mathcal{A}$ as a GWA according to (39). Let X denote the set of positive powers of x. Define $r_n \in \mathcal{A}$ for $n \geq 1$ as in (45). Also define s_j^n, $\mathcal{J}_m^n$, and π_m^n for $n \geq 1$ and $j, m \in \mathbb{Z}$ as in (48), but with everything taking place in $\mathcal{A}$. Define the following subsets of $\operatorname{spec}(\mathcal{A})$:

$$\begin{aligned}
T_1 &= \{P \in \operatorname{spec}(\mathcal{A}) \mid u \in P\}, \\
T_2 &= \{P \in \operatorname{spec}(\mathcal{A}) \mid P = \langle P \cap k[t,d] \rangle\}, \text{ and} \\
T_{3n} &= \{P \in \operatorname{spec}(\mathcal{A}) \mid u, t \notin P,\ P \cap X = \{x^n, x^{n+1}, \ldots\}\} \quad \text{for } n \geq 1.
\end{aligned}$$

Theorem 105: *The prime spectrum of $\mathcal{A}$ is, as a set, the disjoint union of T_1, T_2, and T_{3n} for $n \geq 1$. Each of these subsets is homeomorphic to the prime spectrum of a commutative algebra as follows:*

- $\operatorname{spec}(k[u_{11}, u_{12}, u_{21}]) \approx T_1$ *via* $\mathfrak{p} \mapsto \langle u \rangle + \langle \mathfrak{p} \rangle$.

- $\operatorname{spec}(k[t,d]) \approx T_2$ *via* $\mathfrak{p} \mapsto \langle \mathfrak{p} \rangle$.
- $\operatorname{spec}(k[t^{\pm}]) \approx T_{3n}$ *via* $\mathfrak{p} \mapsto \langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + \langle r_n \rangle + \langle \mathfrak{p} \cap k[t] \rangle$, *for all* $n \geq 1$.

Our proof will make use of the localizations and quotients of $\mathcal{A}$ that were described in the introduction to section 3.1.4. Many of them are quantum tori, so it will help that the prime spectrum of a quantum torus is known.

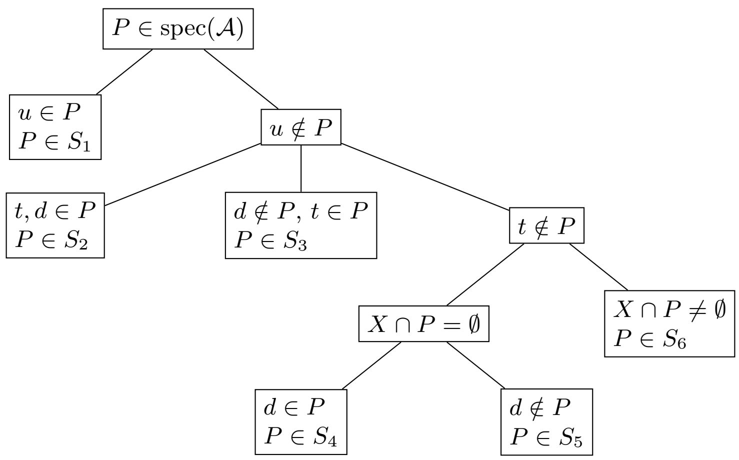
Definition 106: A *quantum torus* over a field k is an iterated skew Laurent algebra

$$k[x_1^{\pm}][x_2^{\pm}; \tau_2] \cdots [x_n^{\pm}; \tau_n]$$

for some $n \in \mathbb{Z}_{\geq 0}$ and some automorphisms $\tau_2, \ldots, \tau_n$ such that $\tau_i(x_j)$ is a nonzero scalar multiple of x_j for all $i \in \{2, \ldots, n\}$ and $j \in \{1, \ldots, i-1\}$.

Lemma 107: [20, Corollary 1.5b] *Contraction and extension provide mutually inverse homeomorphisms between the prime spectrum of a quantum torus and the prime spectrum of its center.*

Proof of Theorem 105: Consider the partition of $\operatorname{spec}(\mathcal{A})$ into subsets $S_1, \ldots, S_6$ given by the following tree, in which branches represent mutually exclusive possibilities:



It is easy to verify that

$$S_1 = T_1,$$
$$S_6 = \bigsqcup_{n \geq 1} T_{3n}.$$

To establish that $\{T_1, T_2\} \cup \{T_{3n} \mid n \geq 1\}$ is a partition of $\operatorname{spec}(\mathcal{A})$, we will show that

$$S_2 \cup S_3 \cup S_4 \cup S_5 = T_2. \tag{88}$$

Let $P \in T_2$ and let $\mathfrak{p} = P \cap k[t,d]$. Then, using the same reasoning as in (14), $P_m = \mathfrak{p}\, k[u,t,d]$ for all $m \in \mathbb{Z}$. In particular, $u \notin P$, so $P \notin S_1$, and $P_n = P_0$ for all $n \geq 1$, so $P \notin S_6$. This establishes the inclusion $\supseteq$ of (88). We now address the reverse inclusion.

$S_2 \subseteq T_2$: Since u is normal, a prime ideal of $\mathcal{A}$ that excludes u also excludes any power of u. So $S_2 \approx \operatorname{spec}(\mathcal{A}_u/\langle t, d \rangle)$. Since $\mathcal{A}_u/\langle t, d \rangle = k[u^{\pm}][x^{\pm}; \sigma]$ is a quantum torus, Lemma 107 and Proposition 9 give that $S_2 \approx \operatorname{spec}(k)$. Let $\mathfrak{p}$ be the single point of $\operatorname{spec}(k)$. It corresponds to the zero ideal of

$k[u^\pm][x^\pm;\sigma]$, which corresponds to $\langle t,d\rangle \lhd \mathcal{A}_u$. Now $\langle t,d\rangle \lhd \mathcal{A}$ is prime because $\mathcal{A}/\langle t,d\rangle = k[u][x,y;\sigma,z]$ and $z \notin \langle t,d\rangle$. So by Lemma 143 (appendix), $\mathfrak{p}$ corresponds to $\langle t,d\rangle \lhd \mathcal{A}$, and we have

$$S_2 = \{\langle t,d\rangle\} \subseteq T_2.$$

$S_3 \subseteq T_2$**:** Since d is central, a prime ideal of $\mathcal{A}$ that excludes d also excludes any power of d. So $S_3 \approx \operatorname{spec}(\mathcal{A}_{ud}/\langle t\rangle)$.

> **Claim:** In the algebra $\mathcal{A}_{ud}/\langle t\rangle = k[u^\pm,d^\pm][x,y;\sigma,z]$, one has $\langle x^n\rangle = \langle 1\rangle$ for all $n \in \mathbb{Z}_{\geq 0}$.
> **Proof:** Let $n \geq 1$. Multiplying x^n by y on either side shows that $\langle x^n\rangle_{n-1}$ contains z and $\sigma^n(z)$. Here these are $d - q^{-4}u^2$ and $d - q^{4n-4}u^2$. Since u is invertible and q is not a root of unity, this implies that $\langle x^n\rangle_{n-1} = \langle 1\rangle$; i.e. $x^{n-1} \in \langle x^n\rangle$. This works for all $n \geq 1$, so we conclude by induction that $1 \in \langle x^n\rangle$.

Thus all prime ideals of $k[u^\pm,d^\pm][x,y;\sigma,z]$ are disjoint from the set of powers of x. Therefore by localization, using Proposition 21 and Theorem 142, $\operatorname{spec}(k[u^\pm,d^\pm][x,y;\sigma,z]) \approx \operatorname{spec}(k[u^\pm,d^\pm][x^\pm;\sigma])$. Lemma 107 and Proposition 9 give that $S_3 \approx \operatorname{spec}(k[d^\pm])$. Let us start with a $\mathfrak{p} \in \operatorname{spec}(k[d^\pm])$ and follow it back to S_3: $\mathfrak{p}$ corresponds to its extension $\langle\mathfrak{p}\rangle \lhd k[u^\pm,d^\pm][x^\pm;\sigma]$. Now $\langle\mathfrak{p}\rangle \lhd \mathcal{A}_{ud}/\langle t\rangle$ is prime because the quotient by it is $(k[u^\pm,d^\pm]/\langle\mathfrak{p}\rangle)[x,y;\sigma,z]$, a GWA over a domain with $z \neq 0$. Hence by Lemma 143, $\mathfrak{p}$ corresponds to $\langle\mathfrak{p}\rangle \lhd \mathcal{A}_{ud}/\langle t\rangle$. This in turn corresponds to $\langle t\rangle + \langle\mathfrak{p}\rangle = \langle t\rangle + \langle\mathfrak{p}\cap k[d]\rangle \lhd \mathcal{A}_{ud}$. Now $\langle t\rangle + \langle\mathfrak{p}\cap k[d]\rangle \lhd \mathcal{A}$ is prime because the quotient by it is $(k[u,d]/\langle\mathfrak{p}\cap k[d]\rangle)[x,y;\sigma,z]$, a GWA over a domain with $z \neq 0$. Hence by Lemma 143, $\mathfrak{p}$ corresponds to $\langle t\rangle + \langle\mathfrak{p}\cap k[d]\rangle \lhd \mathcal{A}$. So

$$S_3 = \{\langle t\rangle + \langle\mathfrak{p}\cap k[d]\rangle \mid \mathfrak{p} \in \operatorname{spec}(k[d^\pm])\} \subseteq T_2.$$

$S_4 \subseteq T_2$**:** Since t is central, $S_4 \approx \operatorname{spec}(\mathcal{A}_{utx}/\langle d\rangle)$. Since $\mathcal{A}_{utx}/\langle d\rangle = k[u^\pm,t^\pm][x^\pm;\sigma]$ is a quantum torus, Lemma 107 and Proposition 9 give that $S_4 \approx \operatorname{spec}(k[t^\pm])$. Let $\mathfrak{p} \in \operatorname{spec}(k[t^\pm])$, and let us follow $\mathfrak{p}$ back to S_4: $\mathfrak{p}$ corresponds to its extension $\langle\mathfrak{p}\rangle \lhd \mathcal{A}_{utx}/\langle d\rangle$, which in turn corresponds to $\langle d\rangle + \langle\mathfrak{p}\rangle = \langle d\rangle + \langle\mathfrak{p}\cap k[t]\rangle \lhd \mathcal{A}_{utx}$. Now $\langle d\rangle + \langle\mathfrak{p}\cap k[t]\rangle \lhd \mathcal{A}$ is prime because the quotient by it is $(k[u,t]/\langle\mathfrak{p}\cap k[t]\rangle)[x,y;\sigma,z]$, a GWA over a domain with $z \neq 0$. Hence by Lemma 143, $\mathfrak{p}$ corresponds to $\langle d\rangle + \langle\mathfrak{p}\cap k[t]\rangle \lhd \mathcal{A}$. So

$$S_4 = \{\langle d\rangle + \langle\mathfrak{p}\cap k[t]\rangle \mid \mathfrak{p} \in \operatorname{spec}(k[t^\pm])\} \subseteq T_2.$$

$S_5 \subseteq T_2$**:** We have $S_5 \approx \operatorname{spec}(\mathcal{A}_{utxd})$. Since $\mathcal{A}_{utxd} = k[u^\pm,t^\pm,d^\pm][x^\pm;\sigma]$ is a quantum torus, Lemma 107 and Proposition 9 give $S_5 \approx k[t^\pm,d^\pm]$. Let $\mathfrak{p} \in \operatorname{spec}(k[t^\pm,d^\pm])$, and let us follow it back to S_5: $\mathfrak{p}$ corresponds to its extension $\langle\mathfrak{p}\rangle = \langle\mathfrak{p}\cap k[t,d]\rangle \lhd \mathcal{A}_{utxd}$. Now $\langle\mathfrak{p}\cap k[t,d]\rangle \lhd \mathcal{A}$ is prime because the quotient by it is $(k[u,t,d]/\langle\mathfrak{p}\cap k[t,d]\rangle)[x,y;\sigma,z]$, a GWA over a domain with $z \neq 0$. Hence by Lemma 143, $\mathfrak{p}$ corresponds to $\langle\mathfrak{p}\cap k[t,d]\rangle \lhd \mathcal{A}$. So

$$S_5 = \{\langle\mathfrak{p}\cap k[t,d]\rangle \mid \mathfrak{p} \in \operatorname{spec}(k[t^\pm,d^\pm])\} \subseteq T_2.$$

We have established (88), proving that

$$\operatorname{spec}(\mathcal{A}) = T_1 \sqcup T_2 \sqcup \bigsqcup_{n\geq 1} T_{3n}.$$

The remainder of the proof establishes homeomorphisms of T_1, T_2, and the T_{3n} to spectra of commutative algebras.

T_1**:** Clearly, T_1 is homeomorphic to the prime spectrum of $\mathcal{A}/\langle u\rangle \cong k[u_{11},u_{12},u_{21}]$ via $\mathfrak{p} \mapsto \langle u\rangle + \langle\mathfrak{p}\rangle$.

T_2**:** Note that the ring extension $k[u,t,d]^\sigma = k[t,d] \subseteq k[u,t,d]$ satisfies the condition (11). It also satisfies the condition (13), due to Proposition 19. We may therefore apply Lemma 20 to conclude that $T_2 \approx \operatorname{spec}(k[t,d])$, with $\mathfrak{p} \in \operatorname{spec}(k[t,d])$ corresponding to $\langle\mathfrak{p}\rangle \lhd \mathcal{A}$.

T_{3n}**:** Let $n \geq 1$. We have $T_{3n} \approx \operatorname{spec}(A_{(n)}/\langle x^n \rangle)$. By Corollary 104, we in turn have $\operatorname{spec}(A_{(n)}/\langle x^n \rangle) \approx \operatorname{spec}(k[t^{\pm}])$. Let $\mathfrak{p} \in \operatorname{spec}(k[t^{\pm}])$, and let us follow it back to T_{3n}. In Corollary 104, $\mathfrak{p}$ corresponds to

$$\bigoplus_{m \in \mathbb{Z}} (\langle \pi_m^n \rangle + \langle \mathfrak{p} \rangle) v_m \ \triangleleft\ \mathcal{A}_{ut}/\langle r_n \rangle,$$

which pulls back to

$$\bigoplus_{m \in \mathbb{Z}} (\langle \pi_m^n, r_n \rangle + \langle \mathfrak{p} \rangle) v_m \ \triangleleft\ \mathcal{A}_{ut}. \tag{89}$$

Applying Lemma 143 to pull back further is not trivial this time, so we instead reference Lemma 24 to see that the pullback of (89) is

$$\bigoplus_{m \in \mathbb{Z}} (\langle \pi_m^n, r_n \rangle + \langle \mathfrak{p} \cap k[t] \rangle) v_m \ \triangleleft\ \mathcal{A},$$

which is another way of writing

$$\langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + \langle r_n \rangle + \langle \mathfrak{p} \cap k[t] \rangle \ \triangleleft\ \mathcal{A}.$$

■

With this description of the prime ideals, we can answer some questions about the algebra $\mathcal{A}$. The notion of a *noetherian UFD* was introduced in [10]. A ring A is said to be a *noetherian UFD* if it is a noetherian domain in which every nonzero prime ideal contains a nonzero principal prime ideal, and in which every height one prime ideal is completely prime.

Corollary 108: *The algebra $\mathcal{A}$ is a noetherian UFD.*

Proof: Having just listed all the prime ideals of $\mathcal{A}$, we simply check off the needed conditions:

- $\mathcal{A}$ is a noetherian domain.
- Every nonzero prime ideal of $\mathcal{A}$ contains a nonzero principal prime ideal. (Here a *principal* ideal is one generated by a single normal element). *Proof:* For T_1 and T_2 this is obvious. For $P \in T_{3n}$, $n \geq 1$, note that P contains $\langle r_n \rangle \in T_2$.
- Height one primes of $\mathcal{A}$ are completely prime. *Proof:* Since $\langle r_n \rangle$ is properly contained in any $P \in T_{3n}$ for $n \geq 1$, the primes in T_{3n} are not height one. We check that all the other primes are completely prime. Suppose $P \in T_2$. Then P is generated in the commutative coefficient ring $k[u,t,d]$ of the GWA $\mathcal{A} = k[u,t,d][x,y;\sigma,z]$ and it does not contain z, so Proposition 12 shows that $\mathcal{A}/P$ is a GWA over a domain, and hence a domain. For $P = \langle u \rangle + \langle \mathfrak{p} \rangle \in T_1$, $\mathcal{A}/P$ is $k[u_{11}, u_{12}, u_{21}]/\mathfrak{p}$, which is a domain.

■

Since $\mathcal{A}$ is noetherian, every closed subset of $\operatorname{spec}(\mathcal{A})$ is a finite union of irreducible closed subsets. The topology of $\operatorname{spec}(\mathcal{A})$ is therefore known if all inclusions of prime ideals are known. We address in the following proposition those inclusions that are not already expressed in Theorem 105.

Proposition 109: *The inclusions among the prime ideals of $\mathcal{A}$ are as follows:*

1. *Inclusions coming from the homeomorphisms $T_1 \approx \operatorname{spec}(k[u_{11}, u_{12}, u_{21}])$, $T_2 \approx \operatorname{spec}(k[t,d])$, and $T_{3n} \approx \operatorname{spec}(k[t^{\pm}])$ for $n \geq 1$.*
2. *Let $P \in T_1$. No prime in T_2 contains P, and no prime in T_{3n} contains P for any n.*
3. *Let $P \in T_2$, say $P = \langle \mathfrak{p} \rangle$ with $\mathfrak{p} \in \operatorname{spec}(k[t,d])$.*

(a) *The set of primes in T_1 that contain P is*

$$\{\langle u\rangle + \langle \mathfrak{q}\rangle \mid \mathfrak{q} \in \operatorname{spec}(k[u_{11}, u_{12}, u_{21}]) \text{ and } \mathfrak{p} \subseteq \phi^{-1}(\mathfrak{q})\},$$

where ϕ is the homomorphism $\phi : k[t, d] \to k[u_{11}, u_{12}, u_{21}]$ that sends t to u_{11} and d to $u_{12}u_{21}$.

(b) *Let $n \geq 1$. The set of primes in T_{3n} that contain P is*

$$\{\langle \pi^n_m v_m \mid -n \leq m \leq n\rangle + \langle r_n\rangle + \langle \mathfrak{q} \cap k[t]\rangle \mid \mathfrak{q} \in \operatorname{spec}(k[t^{\pm}]) \text{ and } \mathfrak{p} \subseteq \eta_n^{-1}(\mathfrak{q})\},$$

where η_n is the homomorphism $\eta_n : k[t^{\pm}, d] \to k[t^{\pm}]$ that sends t to t and d to $\frac{-q^{2n}}{(q^{2n}+1)^2}t^2$.

4. *Let $n \geq 1$ and let $P \in T_{3n}$, say*

$$P = \langle \pi^n_m v_m \mid -n \leq m \leq n\rangle + \langle r_n\rangle + \langle \mathfrak{p} \cap k[t]\rangle$$

with $\mathfrak{p} \in \operatorname{spec}(k[t^{\pm}])$. If $\mathfrak{p} = 0$, then the only prime in T_1 containing P is

$$\langle u_{11}, u_{22}, u_{21}, u_{12}\rangle.$$

If $\mathfrak{p} \neq 0$, then no prime in T_1 or T_2 contains P, and no prime in $T_{3n'}$ contains P for any $n' \neq n$.

Proof: The inclusions of assertion 1 are addressed by the homeomorphisms in Theorem 105.

2: If $P \in T_1$, then $u \in P$. If $Q \in T_2$ then Q_0 (using the notation of Definition 13) is generated in $k[u, t, d]$ by elements of $k[t, d]$, so Q cannot contain u and therefore cannot contain P. If $Q \in T_{3n}$, then by definition Q cannot contain u and therefore cannot contain P.

3a: Assume the setup of assertion 3a. Suppose that $Q \in T_1$, and write it as $\langle u\rangle + \langle \mathfrak{q}\rangle$ with $\mathfrak{q} \in \operatorname{spec}(k[u_{11}, u_{12}, u_{21}])$. Then $P \subseteq Q$ if and only if $\langle u\rangle + \langle \mathfrak{p}\rangle \subseteq Q$, which holds if and only if $(\langle u\rangle + \langle \mathfrak{p}\rangle)/\langle u\rangle \subseteq Q/\langle u\rangle$ holds in $\mathcal{A}/\langle u\rangle$. The following composite is the homomorphism ϕ that we defined:

$$\begin{array}{ccccccc} k[t,d] & \hookrightarrow & \mathcal{A} & \twoheadrightarrow & \mathcal{A}/\langle u\rangle & \cong & k[u_{11}, u_{12}, u_{21}] \\ \mathfrak{p} & & Q & \mapsto & Q/\langle u\rangle & \leftrightarrow & \mathfrak{q}. \end{array}$$

We see that $P \subseteq Q$ if and only if $\phi(\mathfrak{p}) \subseteq \mathfrak{q}$. This holds if and only if $\mathfrak{p} \subseteq \phi^{-1}(\mathfrak{q})$, so assertion 3a is proven.

3b: Assume the setup of assertion 3b. Suppose that $Q \in T_{3n}$, and write it as

$$\langle \pi^n_m v_m \mid -n \leq m \leq n\rangle + \langle r_n\rangle + \langle \mathfrak{q} \cap k[t]\rangle$$

with $\mathfrak{q} \in \operatorname{spec}(k[t^{\pm}])$. Then

$$Q = \bigoplus_{m \in \mathbb{Z}} (\langle \pi^n_m, r_n\rangle + \langle \mathfrak{q} \cap k[t]\rangle)v_m;$$

the inclusion $\supseteq$ is clear and the inclusion $\subseteq$ follows from the fact that the right hand side is an ideal of $\mathcal{A}$, which can be verified by using (49) and (50) to check that the conditions of Proposition 18 are met. In particular, $Q_0 = \langle \pi^n_0, r_n\rangle + \langle \mathfrak{q} \cap k[t]\rangle$. Now assertion 3b is proven as follows:

$$\begin{aligned} P \subseteq Q &\Leftrightarrow \mathfrak{p} \subseteq Q_0 = \langle \pi^n_0, r_n\rangle + \langle \mathfrak{q} \cap k[t]\rangle \\ &\Leftrightarrow \mathfrak{p} \subseteq \langle r_n\rangle_{k[t,d]} + \langle \mathfrak{q} \cap k[t]\rangle \qquad (90) \\ &\Leftrightarrow \mathfrak{p} \subseteq \langle r_n\rangle_{k[t^{\pm},d]} + \langle \mathfrak{q}\rangle \qquad (91) \\ &\Leftrightarrow (\langle r_n\rangle_{k[t^{\pm},d]} + \langle \mathfrak{p}\rangle_{k[t^{\pm},d]})/\langle r_n\rangle_{k[t^{\pm},d]} \subseteq (\langle r_n\rangle_{k[t^{\pm},d]} + \langle \mathfrak{q}\rangle)/\langle r_n\rangle_{k[t^{\pm},d]} \\ &\Leftrightarrow \eta_n(\mathfrak{p}) \subseteq \mathfrak{q} \qquad (92) \\ &\Leftrightarrow \mathfrak{p} \subseteq \eta_n^{-1}(\mathfrak{q}). \end{aligned}$$

Line (90) is due to the fact that $Q_0 \cap k[t, d] = \langle r_n\rangle_{k[t,d]} + \langle \mathfrak{q} \cap k[t]\rangle$. Line (91) is due to the fact that $k[t, d]$ mod the ideal $\langle r_n\rangle_{k[t,d]} + \langle \mathfrak{q} \cap k[t]\rangle$ is t-torsionfree. Line (92) is due to the fact that η_n is the following composite:

$$\begin{array}{ccccc} k[t^{\pm}, d] & \twoheadrightarrow & k[t^{\pm}, d]/\langle r_n\rangle & \cong & k[t^{\pm}] \\ & & \langle \mathfrak{q}\rangle + \langle r_n\rangle & \leftrightarrow & \mathfrak{q}. \end{array}$$

4: Assume the setup of assertion 4. Let $Q \in T_1$ such that $P \subseteq Q$, say $Q = \langle u_{22} \rangle + \langle \mathfrak{q} \rangle$ with $\mathfrak{q} \in \mathrm{spec}(k[u_{11}, u_{12}, u_{21}])$. Then Q contains a power of x and a power of y, so $\mathfrak{q}$ contains u_{21} and u_{12}. Q also contains r_n, which is equivalent to $q^{2n} u_{11}$ modulo $\langle u_{22}, u_{12}, u_{21} \rangle$. So Q contains, and therefore equals, the maximal ideal $\langle u_{11}, u_{22}, u_{21}, u_{12} \rangle$. The containment $P \subseteq \langle u_{11}, u_{22}, u_{21}, u_{12} \rangle$ clearly holds if $\mathfrak{p} = 0$. But if $\mathfrak{p}$ is nonzero, then it contains some polynomial in t with nonzero constant term, which is not in $\langle u_{11}, u_{22}, u_{21}, u_{12} \rangle$. Thus, nothing in T_1 contains P when $\mathfrak{p} \neq 0$.

If $Q \in T_2$, then $Q = \bigoplus_{m \in \mathbb{Z}} Q_0 v_m$ does not contain any power of x. So nothing in T_2 contains P.

Now suppose that $Q \in T_{3n'}$ with $n' \neq n$, and suppose for the sake of contradiction that $P \subseteq Q$. Then Q contains r_n and $r_{n'}$. Since $n \neq n'$, it follows that $t, d \in Q$. Write Q as

$$Q = \left\langle \pi_m^{n'} v_m \mid -n' \leq m \leq n' \right\rangle + \langle r_{n'} \rangle + \langle \mathfrak{q} \cap k[t] \rangle$$

with $\mathfrak{q} \in \mathrm{spec}(k[t^{\pm}])$. Since $t \in Q$, we must have $\mathfrak{q} = 0$. We have a contradiction:

$$d \in Q_0 = \left\langle \pi_0^{n'}, r_{n'} \right\rangle \lhd k[u, t, d]. \qquad \blacksquare$$

The prime spectrum of a ring A is said to satisfy *normal separation* if for every pair of distinct comparable primes $P \subsetneq Q$, the ideal Q/P of A/P contains a nonzero normal element. A ring is said to be *catenary* when for every pair of prime ideals $P \subseteq Q$, all saturated chains of prime ideals between P and Q have the same length. An algebra A is said to satisfy *Tauvel's height formula* if $\mathrm{height}(P) + \mathrm{GK}(A/P) = \mathrm{GK}(A)$ for every prime ideal P of A. These three properties have been shown to be related under certain homological and GK dimension hypotheses; see [8, II.9.5] for details. We will show that all three properties fail to hold for the algebra $\mathcal{A}$.

Proposition 110: *The algebra $\mathcal{A}$ does not have normal separation.*

Proof: Let $P = \langle \pi_0^1, x, y, r_1 \rangle$ and let $Q = \langle r_1 \rangle$, both prime ideals of $\mathcal{A}$. We will show that no element of $P \setminus Q$ is normal modulo Q. Note that $k[u, t, d]/\langle r_1 \rangle \cong k[u, t]$, and let $R = k[u, t]$. Using Proposition 12, $\mathcal{A}/Q$ is isomorphic to

$$W := R[x, y; \sigma, z = -q^{-4} s_1^1 s_2^1],$$

and P/Q becomes

$$\overline{P} := \langle \pi_0^1, x, y \rangle = \bigoplus_{m>0} R y^m \oplus \langle s_1^1 \rangle_R \oplus \bigoplus_{m>0} R x^m.$$

By Proposition 11, the nonzero normal elements of W are the σ-eigenvectors in R. Thus, they are all of the form $u^i f(t)$ for some polynomial $f(t)$ and some $i \in \mathbb{Z}_{\geq 0}$. But $\overline{P}$ cannot contain such elements, since $\overline{P}_0 = \langle s_1^1 \rangle_R$. $\blacksquare$

Proposition 111: *The algebra $\mathcal{A}$ is not catenary.*

Proof: Let $n \geq 1$. The information in Proposition 109 implies that the following two chains of primes are saturated:

$$\begin{tikzcd}[column sep=small]
& \langle u_{11}, u_{22}, u_{21}, u_{12} \rangle \arrow[dl, dash] \arrow[ddr, dash] & \\
\langle u_{11}, u_{22}, u_{21} \rangle \arrow[dd, dash] & & \\
& & \langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + \langle r_n \rangle \arrow[ddl, dash] \\
\langle r_n, u_{22} \rangle \arrow[dr, dash] & & \\
& \langle r_n \rangle &
\end{tikzcd}$$

■

Proposition 112: *The algebra $\mathcal{A}$ does not satisfy Tauvel's height formula.*

Proof: Fix any $n \geq 1$ and consider the prime ideal $P = \langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + \langle r_n \rangle$. The information in Proposition 109 implies that height $P = 2$. Since P contains x^n and y^n, the algebra $\mathcal{A}/P$ is finitely generated as a $k[u, t, d]$-module. Let $\overline{R}$ be the image of $k[u, t, d]$ in $\mathcal{A}/P$. Using [32, 8.2.9(ii)] and [32, 8.2.13], we have

$$\mathrm{GK}(\mathcal{A}/P) = \mathrm{GK}(\overline{R}) = \mathrm{GK}(\overline{R}_u),$$

where the subscript indicates localization. Note that $P_0 = \langle \pi_0^n, r_n \rangle$. We have

$$\overline{R}_u \cong (R/P_0)_u \cong k[u^{\pm}, t]/\langle \pi_0^n \rangle \cong \prod_{j=1}^{n} k[u^{\pm}, t]/\langle s_j^n \rangle \cong k[u^{\pm}]^{\times n}$$

by the Chinese remainder theorem. Viewing the latter product of rings as a direct sum of left regular modules, we may apply [32, 8.3.2(i)] to obtain

$$\mathrm{GK}(\overline{R}_u) = \mathrm{GK}(k[u^{\pm}]) = 1.$$

Hence $\mathrm{GK}(\mathcal{A}/P) = 1$, and we have a violation of the height formula:

$$\mathrm{ht}(P) + \mathrm{GK}(\mathcal{A}/P) = 2 + 1 \neq \mathrm{GK}(\mathcal{A}) = 4.$$ ■

One reason to compute the prime spectrum of an algebra is to make progress towards the lofty goal of knowing its complete representation theory. The idea is to make progress by trying to know the algebra's *primitive ideals*, those ideals that arise as annihilators of irreducible representations. Since primitive ideals are prime, one approach is to determine the prime spectrum of the algebra and then attempt to locate the primitives living in it. The Dixmier-Moeglin equivalence, when it holds, provides a topological criterion for picking out primitives from the spectrum; see [8, II.7-II.8] for definitions.

Theorem 113: *The algebra $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence, and its primitive ideals are as follows:*

- *The primitive ideals in T_1 are $\langle u \rangle + \langle \mathfrak{p} \rangle$ for $\mathfrak{p} \in \max\operatorname{spec} k[u_{11}, u_{12}, u_{21}]$.*
- *The primitive ideals in T_2 are $\langle \mathfrak{p} \rangle$ for $\mathfrak{p} \in \max\operatorname{spec} k[t, d]$.*
- *The primitive ideals in T_{3n} are $\langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + \langle r_n \rangle + \langle \mathfrak{p} \cap k[t] \rangle$ for $n \geq 1$ and $\mathfrak{p} \in \max\operatorname{spec} k[t^{\pm}]$.*

Proof: We first observe, by using Proposition 161, that $\mathcal{A}$ satisfies the Nullstellensatz over k. It then follows from [8, II.7.15] that the following implications hold for all prime ideals of $\mathcal{A}$:

$$\text{locally closed} \implies \text{primitive} \implies \text{rational}.$$

To establish the Dixmier-Moeglin equivalence for $\mathcal{A}$, it remains to close the loop and show that rational primes are locally closed. We shall deal separately with the three different types of primes identified in Theorem 105.

T_1**:** Suppose that $P \in T_1$, say $P = \langle u\rangle + \langle \mathfrak{p}\rangle$ with $\mathfrak{p} \in \operatorname{spec}(k[u_{11}, u_{12}, u_{21}])$. Then $\mathcal{A}/P \cong k[u_{11}, u_{12}, u_{21}]/\mathfrak{p}$. It follows that P is rational if and only if $\mathfrak{p}$ is a maximal ideal of $k[u_{11}, u_{12}, u_{21}]$. In this case P will be maximal and therefore locally closed. Thus, rational primes in T_1 are locally closed.

T_2**:** Suppose that $P \in T_2$, say $P = \langle \mathfrak{p}\rangle$ with $\mathfrak{p} \in \operatorname{spec}(k[t, d])$. Then, using Proposition 12, $\mathcal{A}/P$ is a GWA $R[x, y; \sigma, z]$, where $R := k[u, t, d]/\langle \mathfrak{p}\rangle$. Since $z = d + q^{-2}tu - q^{-4}u^2$ is regular in R, Proposition 5 tells us that $R[x, y; \sigma, z]$ embeds into the skew Laurent polynomial algebra $R[x^{\pm}; \sigma]$. Let K denote the fraction field of R. The skew Laurent polynomial algebra $R[x^{\pm}; \sigma]$ embeds into the skew Laurent *series* algebra $K((x^{\pm}; \sigma))$. (We are abusing notation and writing σ for the induced automorphism of K.) Since the skew Laurent series algebra is a division ring, we obtain an induced embedding of the Goldie quotient ring $\operatorname{Fract}(\mathcal{A}/P)$ into it:

$$\operatorname{Fract}(\mathcal{A}/P) \hookrightarrow K((x^{\pm}; \sigma)).$$

For something to be in the center of $\operatorname{Fract}(\mathcal{A}/P) \cong \operatorname{Fract}(R[x, y; \sigma, z])$, it must at least commute with R and x. This is sufficient to place it in the center of $K((x^{\pm}; \sigma))$, so

$$Z(\operatorname{Fract}(\mathcal{A}/P)) \cong Z(K((x^{\pm}; \sigma))) \cap \operatorname{Fract}(\mathcal{A}/P). \tag{93}$$

According to Proposition 9, the center of $K((x^{\pm}; \sigma))$ is the fixed subfield K^{σ}. Since K is wholly contained in $Z(\operatorname{Fract}(\mathcal{A}/P)) \cong Z(\operatorname{Fract}(R[x, y; \sigma, z]))$, (93) becomes

$$Z(\operatorname{Fract}(\mathcal{A}/P)) \cong K^{\sigma}.$$

Now to compute K^{σ}. Since

$$R = k[u, t, d]/\langle \mathfrak{p}\rangle \cong (k[t, d]/\mathfrak{p})[u],$$

K is the rational function field $L(u)$, where L is the fraction field of $k[t, d]/\mathfrak{p}$.

> **Claim:** $K^{\sigma} = L$.
> **Proof:** Observe that σ fixes L and sends u to q^2u. Consider any nonzero $f/g \in K^{\sigma} = L(u)^{\sigma}$, where $f, g \in L[u]$ are coprime. We have $\sigma(f)g = f\sigma(g)$. Since f and g are coprime, it follows that $f \mid \sigma(f)$. Similarly, since $\sigma(f)$ and $\sigma(g)$ are coprime, $\sigma(f) \mid f$. It follows that $\sigma(f) = \alpha f$ for some $\alpha \in L$. From $\sigma(f)g = f\sigma(g)$ it follows that also $\sigma(g) = \alpha g$. We have an eigenspace decomposition for the action of σ as an L-linear operator on $L[u]$; it is $\bigoplus_{i\geq 0} Lu^i$, with distinct eigenvalues since q is not a root of unity. Since f and g are σ-eigenvectors with the same eigenvalue α, there is some $i \geq 0$ such that $f = f_0u^i$ and $g = g_0u^i$, where $f_0, g_0 \in L$. Thus, $f/g = f_0/g_0 \in L$.

We have found that

$$Z(\operatorname{Fract}(\mathcal{A}/P)) \cong L.$$

The fraction field L of $k[t, d]/\mathfrak{p}$ is algebraic over k if and only if $\mathfrak{p} \lhd k[t, d]$ is maximal. Thus, P is rational if and only if $\mathfrak{p}$ is maximal.

Now assume that P is rational and hence that $\mathfrak{p}$ is maximal. For any $\mathfrak{q} \in \operatorname{spec}(k[t^{\pm}])$ and any $n \geq 1$, define

$$Q_{\mathfrak{q},n} := \langle \pi_m^n v_m \mid -n \leq m \leq n\rangle + \langle r_n\rangle + \langle \mathfrak{q} \cap k[t]\rangle \in T_{3n}.$$

If no $Q_{\mathfrak{q},n}$ contains P, then by using Proposition 109 we can see that $\{P\} = V(P) \cap (\operatorname{spec}(\mathcal{A}) \setminus V(u))$, so P is locally closed. Suppose, on the other hand, that $Q_{\mathfrak{q},n}$ contains P for some $\mathfrak{q} \in \operatorname{spec}(k[t^{\pm}])$ and $n \geq 1$. We claim that this can occur for at most one n.

> **Claim:** If $Q_{\mathfrak{q},n}$ and $Q_{\mathfrak{q}',n'}$ both contain P, then $n = n'$.
> **Proof:** According to assertion 3b of Proposition 109, we have
>
> $$\mathfrak{p} \subseteq \eta_n^{-1}(\mathfrak{q}) \text{ and } \mathfrak{p} \subseteq \eta_{n'}^{-1}(\mathfrak{q}'),$$

where $\eta_n, \eta_{n'}$ are the homomorphisms defined there. Since $\mathfrak{p}$ generates a maximal ideal of $k[t^{\pm}, d]$, this forces

$$\eta_n^{-1}(\mathfrak{q}) = \eta_{n'}^{-1}(\mathfrak{q}').$$

We have

$$d + \frac{q^{2n}}{(q^{2n}+1)^2}t^2,\ d + \frac{q^{2n'}}{(q^{2n'}+1)^2}t^2 \in \eta_n^{-1}(\mathfrak{q}) = \eta_{n'}^{-1}(\mathfrak{q}'),$$

so

$$\left(\frac{q^{2n}}{(q^{2n}+1)^2} - \frac{q^{2n'}}{(q^{2n'}+1)^2}\right)t^2 \in \eta_n^{-1}(\mathfrak{q}) = \eta_{n'}^{-1}(\mathfrak{q}').$$

Since we cannot have $\eta_n^{-1}(\mathfrak{q}) = k[t^{\pm}, d]$, the quantity in parentheses must vanish. This leads to the equation

$$0 = q^{2n}(q^{2n'}+1)^2 - q^{2n'}(q^{2n}+1)^2 = (q^{n'} - q^n)(q^{n'} + q^n)(q^{n+n'} - 1)(q^{n+n'} + 1).$$

Since q is not a root of unity, it follows that $n = n'$.

Hence $\{P\} = V(P) \cap (\mathrm{spec}(\mathcal{A}) \setminus (V(u) \cup V(x^n)))$, and again we see that P is locally closed. Thus we have shown that all rational primes in T_2 are locally closed.

T_{3n}**:** Suppose that $n \geq 1$ and $P \in T_{3n}$, say $P = \langle \pi_m^n v_m \mid -n \leq m \leq n\rangle + \langle r_n \rangle + \langle \mathfrak{p} \cap k[t]\rangle$ with $\mathfrak{p} \in \mathrm{spec}(k[t^{\pm}])$. We will show that if P is rational, then $\mathfrak{p}$ must be maximal. Assume that P is rational, but $\mathfrak{p}$ is not maximal (i.e. $\mathfrak{p} = 0$). Then $t \in Z(\mathrm{Fract}(\mathcal{A}/P))$ is algebraic over k, so for some nonzero polynomial $f(T) \in k[T]$ we must have $f(t) = 0 \in Z(\mathrm{Fract}(\mathcal{A}/P))$. For the element t of $\mathcal{A}$, this means that $f(t) \in P$. Since

$$P = \bigoplus_{m\in\mathbb{Z}} \langle \pi_m^n, r_n\rangle v_m,$$

we have $f(t) \in \langle \pi_0^n, r_n\rangle_{k[u,t,d]}$. Pushing this fact into $k[u,t,d]/\langle r_n\rangle \cong k[u,t]$ gives

$$f(t) \in \langle \pi_0^n \rangle_{k[u,t]}$$

which is clearly false.

Thus we have shown that when P is rational, $\mathfrak{p} \triangleleft k[t^{\pm}]$ must be maximal. Using Proposition 109, we see that in this case $\{P\} = V(P)$. So all rational primes in T_{3n} are locally closed.

We have now shown that all rational prime ideals of $\mathcal{A}$ are locally closed, and we conclude that $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence. Further, we have pinpointed which primes are rational in T_1 and T_2. As for T_{3n}, we have found for $P = \langle r_n\rangle + \langle \mathfrak{p} \cap k[t]\rangle + \langle \pi_m^n v_m \mid -n \leq m \leq n\rangle$ that

$$P \text{ rational} \quad\Longrightarrow\quad \mathfrak{p} \text{ maximal} \quad\Longrightarrow\quad P \text{ locally closed.}$$

Putting this information together and applying the Dixmier-Moeglin equivalence, we conclude that the primitive ideals of $\mathcal{A}$ are as stated in the theorem. ■

There is a classical theorem by Duflo (see [15]) which states that for the universal enveloping algebra of a semisimple Lie algebra, every primitive ideal is the annihilator of some highest weight module. An analogue of this property for GWAs was considered in [33]. We will investigate the extent to which this property holds for $\mathcal{A}$.

Definition 114: Let $W = R[x, y; \sigma, z]$ be a GWA over a commutative k-algebra R. A *highest weight* left W-module is defined to be a weight module ${}_W V$ for which there is some $\mathfrak{m} \in \max\mathrm{spec}\, R$ such that $\mathrm{ann}_{\mathfrak{m}} V$ generates V and is annihilated by x.

Definition 115: Let R be a commutative k-algebra. A GWA $W = R[x, y; \sigma, z]$ is said to *have the Duflo property* when every primitive ideal of W is the annihilator of a simple highest weight left W-module.

By Proposition 3, a GWA over a commutative ring is isomorphic to its own opposite ring (via the map that swaps x and y). So the Duflo property is left-right symmetric, despite the appearance of the word "left" in the definition.

Proposition 116: *The algebra $\mathcal{A}$, viewed as a GWA as in (39), does not have the Duflo property.*

Proof: Let $P = \langle u, t, y, x-1\rangle$, a primitive ideal of $\mathcal{A}$ by Theorem 113. Any nonzero left $\mathcal{A}$-module annihilated by P has x acting as the identity map. Such a module could not be a highest weight module. ■

What seems to be causing problems is the vanishing of u, which breaks down the rigid GWA structure. Let us shift our attention to the localization $\mathcal{A}_u$. Make the definitions in (45) and (48) using the same notation that appears there, except with everything taking place in $\mathcal{A}_u$. We will write the primitive spectrum and then show that the Duflo property almost holds.

Proposition 117: *The algebra $\mathcal{A}_u$ satisfies the Dixmier-Moeglin equivalence and its primitive spectrum is as follows:*

$$\begin{aligned}\operatorname{prim}(\mathcal{A}_u) = &\{\langle \mathfrak{p}\rangle \mid \mathfrak{p} \in \operatorname{max\,spec} k[t,d]\} \sqcup \\ &\bigsqcup_{n\geq 1} \{\langle \mathfrak{p}\cap k[t]\rangle + \langle r_n\rangle + \langle \pi_m^n v_m \mid -n \leq m \leq n\rangle \mid \mathfrak{p} \in \operatorname{max\,spec} k[t^{\pm}]\}.\end{aligned}$$

Proof: By Proposition 161, $\mathcal{A}_u$ satisfies the nullstellensatz over k. Since $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence and $\mathcal{A}_u$ satisfies the nullstellensatz over k, we may apply Proposition 162 to conclude that $\mathcal{A}_u$ satisfies the Dixmier-Moeglin equivalence and that its primitive ideals are simply the extensions of primitive ideals of $\mathcal{A}$ that do not contain u. ■

Proposition 118: *Let $W = R[x, y; \sigma, z]$ be a GWA over a commutative k-algebra R. Let $\mathfrak{m}$ be a maximal ideal of R with infinite σ-orbit. If $n(\mathfrak{m}) < \infty$, then $V_{\mathfrak{m}}$ is a simple highest weight module.*

Proof: Of course $V_{\mathfrak{m}}$ is a simple left W-module by its construction. This makes it automatically a weight module, as observed at the beginning of section 2.5.2. Let e_i denote the image of v_i in $V_{\mathfrak{m}}$ for $i \in \mathbb{Z}$. Now

$$V_{\mathfrak{m}} = \bigoplus_{-n'(\mathfrak{m})<i<n(\mathfrak{m})} Re_i.$$

If $n(\mathfrak{m}) < \infty$, then $\operatorname{ann}_{\sigma^{n(\mathfrak{m})-1}(\mathfrak{m})} V_{\mathfrak{m}} = Re_{n(\mathfrak{m})-1}$ is nonzero and it is annihilated by x. ■

Theorem 119: *The algebra $\mathcal{A}_u$ almost has the Duflo property, but fails only due to the primitive ideal $\langle t, d\rangle$. That is,*

$$\operatorname{prim}(\mathcal{A}_u) \setminus \langle t, d\rangle = \{\textit{annihilators of simple highest weight } \mathcal{A}_u\textit{-modules}\}.$$

Proof: From Proposition 117 we know all the primitive ideals. Consider a primitive ideal P of the form $P = \langle \mathfrak{p}\rangle$, where $\mathfrak{p} \in \operatorname{max\,spec} k[t,d]$. Assume that $\mathfrak{p} \neq \langle t, d\rangle$. Consider the obvious isomorphism $k[u^{\pm}, t, d]/\langle \mathfrak{p}\rangle \cong K[u^{\pm}]$, where K is the field $k[t,d]/\mathfrak{p}$. Define $\overline{A} := \mathcal{A}_u/P = K[u^{\pm}][x, y; \sigma, z]$. Note that if we can construct a simple faithful highest weight left $\overline{A}$-module, then as an $\mathcal{A}_u$-module it would be a simple highest weight module with annihilator P.

Claim: For some $\mathfrak{m} \in \operatorname{max\,spec} K[u^{\pm}]$, we have $n(\mathfrak{m}) = 1$ and $n'(\mathfrak{m}) = \infty$.
Proof: Suppose to the contrary that all $\mathfrak{m} \in \operatorname{max\,spec} K[u^{\pm}]$ for which $n(\mathfrak{m}) = 1$ (i.e. for which $z \in \mathfrak{m}$) have $n'(\mathfrak{m}) < \infty$. Since z is not a unit in $K[u^{\pm}]$, there is at least some $\mathfrak{m}$ for which $n(\mathfrak{m}) = 1$.

Fix such an $\mathfrak{m}$. Note that z can only be in finitely many maximal ideals (at most two, in fact). Since $\mathfrak{m}$ has infinite σ-orbit (Proposition 82), there is some minimum $\ell \in \mathbb{Z}$ such that $z \in \sigma^{\ell}(\mathfrak{m})$. By our supposition, we have $n' := n'(\sigma^{\ell}(\mathfrak{m})) < \infty$. Thus $\sigma^{n'}(z) \in \sigma^{\ell}(\mathfrak{m})$, contradicting the minimality of ℓ.

Let $\mathfrak{m}$ be as in the claim. Since $n(\mathfrak{m}) < \infty$, the simple module $V_{\mathfrak{m}}$ is highest weight by Proposition 118 (as noted in the claim, Proposition 82 tells us that $\mathfrak{m}$ has infinite σ-orbit). Proposition 66 provides a description of the annihilator of $V_{\mathfrak{m}}$. Considering that $n'(\mathfrak{m}) = \infty$ and that an infinite intersection of maximal ideals in a PID is zero, we see that $V_{\mathfrak{m}}$ is faithful. Thus P is the annihilator of a simple highest weight $\mathcal{A}_u$-module.

Now let P be a primitive ideal of the form $P = \langle \mathfrak{p} \cap k[t] \rangle + \langle r_n \rangle + \langle \pi^n_m v_m \mid -n \leq m \leq n \rangle$, where $\mathfrak{p} \in \max\operatorname{spec} k[t^{\pm}]$ and $n \geq 1$. Using Proposition 12, we have that $\mathcal{A}_u / (\langle \mathfrak{p} \cap k[t] \rangle + \langle r_n \rangle)$ is isomorphic to $\overline{A} := K[u^{\pm}][x, y; \sigma, z = -q^{-4} s^n_{n+1} s^n_1]$, where K is the field $k[t]/\langle \mathfrak{p} \cap k[t] \rangle \cong k[t^{\pm}]/\mathfrak{p}$ and where we are reusing the notation s^n_j to refer to the images of the s^n_j in $K[u^{\pm}]$. We must find a simple highest weight left $\overline{A}$-module with annihilator $\bigoplus_{m \in \mathbb{Z}} \langle \pi^n_m \rangle_{K[u^{\pm}]} v_m = \langle \pi^n_m v_m \mid -n \leq m \leq n \rangle \lhd \overline{A}$. Let $\mathfrak{m} = \langle s^n_n \rangle$, a maximal ideal of $K[u^{\pm}]$ with infinite σ-orbit. Given the observations (49) and (50) it is easy to see that $n'(\mathfrak{m}) = 1$ and $n(\mathfrak{m}) = n$. Using the Chinese remainder theorem,

$$\begin{aligned} \langle \pi^n_m \rangle &= \left\langle \prod_{j \in \mathcal{J}^n_m} s^n_j \right\rangle = \bigcap_{j \in \mathcal{J}^n_m} \langle s^n_j \rangle = \bigcap_{j \in \mathcal{J}^n_m} \sigma^{n-j}(\mathfrak{m}) = \bigcap \{ \sigma^j(\mathfrak{m}) \mid n - j \in \mathcal{J}^n_m \} \\ &= \bigcap \{ \sigma^j(\mathfrak{m}) \mid 0 \leq j < n \text{ and } m \leq j < n + m \} \end{aligned}$$

for $m \in \mathbb{Z}$. This matches the description of $J(\mathfrak{m})_m$ in Proposition 66, so $V_{\mathfrak{m}}$ has annihilator $\bigoplus_{m \in \mathbb{Z}} \langle \pi^n_m \rangle v_m$, as an $\overline{A}$-module. It follows that P is the annihilator of $V_{\mathfrak{m}}$ as an $\mathcal{A}_u$-module, which is a highest weight module because $n(\mathfrak{m}) = n$ (Proposition 118).

We now argue that $\langle t, d \rangle_{\mathcal{A}_u}$ cannot be the annihilator of a highest weight module. If it were, then one would have a nonzero highest weight module over $\mathcal{A}_u / \langle t, d \rangle \cong k[u^{\pm}][x, y; \sigma, z = -q^{-4} u^2]$. In such a module, there is a nonzero element annihilated by x, and hence by $yx = z$. This is not possible because z is a unit. ■

3.1.5 Prime Spectrum - New Approach

We demonstrate here how the laborious part of section 3.1.4 can be greatly simplified by using the theory of section 2.6.

We first get some notation and calculation in place. Let $\mathcal{A}_u$ again denote the 2×2 REA localized at the set of powers of u, a GWA $k[t, d, u^{\pm}][x, y; \sigma, z]$ as before. For $n > 0$ and $m \in \mathbb{Z}$, define π^n_m as in (48). For convenience in this section, we write out π^n_m in the following format:

$$\pi^n_m = \prod \left\{ \left(u - \frac{q^{-2j}}{q^{-2n} + 1} t \right) \mid 0 \leq j < n \text{ and } m \leq j < n + m \right\}.$$

For $n > 0$ and $\mathfrak{t} \in \operatorname{spec}(k[t^{\pm}])$, define

$$P(n, \mathfrak{t}) = \langle \pi^n_m v_m \mid m \in \mathbb{Z} \rangle + \langle r_n \rangle + \langle \mathfrak{t} \cap k[t] \rangle \lhd \mathcal{A}_u,$$

where r_n is defined just as in (45).

Proposition 120: *Let n be a positive integer. Assume that A is a k-algebra and $d, t, u \in A$ with u being a unit. Then the ideal of A generated by $d + tu - u^2$ and $d + q^{-2n} tu - q^{-4n} u^2$ contains*

$$(q^{2n} + 1)^2 d + q^{2n} t^2.$$

Proof: Define $z_1 = d + tu - u^2$, $z_2 = d + q^{-2n}tu - q^{-4n}u^2$, and $I = \langle z_1, z_2\rangle$. Now $I \ni u^{-1}(z_1 - z_2) = (1-q^{-2n})t - (1-q^{-4n})u$, so $u \equiv (1+q^{-2n})^{-1}t$ modulo I. Substituting for u in z_1 then gives $I \ni d + ((1+q^{-2n})^{-1} - (1+q^{-2n})^{-2})t^2 = d + \frac{q^{2n}}{(q^{2n}+1)^2}t$. ■

Theorem 121: *As a set,* $\operatorname{spec}(\mathcal{A}_u) = T_2 \cup T_3$*, where* $T_2 = \{\langle \mathfrak{p}\rangle \mid \mathfrak{p} \in \operatorname{spec}(k[t,d])\}$ *and*

$$T_3 = \left\{P(n,\mathfrak{t}) \mid n \geq 1 \text{ and } \mathfrak{t} \in \operatorname{spec}(k[t^{\pm}])\right\}.$$

Proof: Since q is not a root of unity, the linear operator $a \mapsto u^{-1}au$ has the graded components of $\mathcal{A}_u$ as its distinct eigenspaces. It follows that every ideal of $\mathcal{A}_u$ is homogeneous (Proposition 81). Let $Z = k[t,d]$.

Since Z is central, the contraction $P \cap Z$ of any prime P of $\mathcal{A}_u$ is a prime ideal of Z. Thus

$$\operatorname{spec}(\mathcal{A}_u) = \bigcup_{\mathfrak{p}\in\operatorname{spec}(Z)} F_{\mathfrak{p}},$$

where $F_{\mathfrak{p}} := \{P \in \operatorname{spec}(\mathcal{A}_u) \mid P \cap Z = \mathfrak{p}\} = \{P \in \text{gr-spec}(\mathcal{A}_u) \mid P \cap Z = \mathfrak{p}\}$.

Fix a $\mathfrak{p} \in \operatorname{spec}(Z)$. Define $Z' = Z/\mathfrak{p}$ and let Z'' denote the fraction field of Z'. We identify $A' := \mathcal{A}_u/\langle\mathfrak{p}\rangle$ with $Z'[u^{\pm}][x,y;\sigma,z]$ using Proposition 12. We identify the localization A'' of A' at $Z' \setminus \{0\}$ with $Z''[u^{\pm}][x,y;\sigma,z]$, using Proposition 22. Now $F_{\mathfrak{p}}$ is in bijection with $\text{gr-spec}(A'')$ via pullback along the quotient and localization maps $A \to A' \to A''$.

By Proposition 82 and the fact that q is not a root of unity, every maximal ideal of $Z''[u^{\pm}]$ has infinite σ-orbit. Thus we may apply Corollary 71 to A''. (To apply these things, view A'' as an algebra over the field Z'', so that the base ring $Z''[u^{\pm}]$ is an affine algebra). This gives $\text{gr-spec}(A'') = \{0\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}\}$, where $\mathscr{M}'_{\mathrm{II}} = \{\mathfrak{m} \in \max\operatorname{spec}(Z''[u^{\pm}]) \mid \sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m} \text{ for some } n > 0\}$.

If $\mathscr{M}'_{\mathrm{II}} \neq \emptyset$, then Proposition 120 tells us that, for some $n > 0$, the image of r_n lies in some maximal ideal of $Z''[u^{\pm}]$. Since also $r_n \in Z$, it would follow that r_n vanishes in the field Z''. In other words, if $\mathscr{M}'_{\mathrm{II}} \neq \emptyset$, then $r_n \in \mathfrak{p}$ for some $n > 0$. So if $r_n \notin \mathfrak{p}$ for all $n > 0$, then $F_{\mathfrak{p}} = \{\langle\mathfrak{p}\rangle\}$.

If $r_n \in \mathfrak{p}$ and also $t \in \mathfrak{p}$, then $d \in \mathfrak{p}$ and $\mathfrak{p} = \langle t, d\rangle_Z$. In this case, z becomes a unit in $Z''[u^{\pm}]$. Then obviously $\mathscr{M}'_{\mathrm{II}} = \emptyset$ and we see that $F_{\langle t,d\rangle} = \{\langle t,d\rangle\}$.

Suppose now that $r_n \in \mathfrak{p}$ for some $n > 0$, and that $t \notin \mathfrak{p}$. Using the fact that $Z/\langle r_n\rangle$ is isomorphic to $k[t]$ by an isomorphism that fixes t, we obtain the following bijective correspondence:

$$\begin{array}{ccc} \{\mathfrak{p} \in \operatorname{spec}(Z) \mid r_n \in \mathfrak{p} \text{ and } t \notin \mathfrak{p}\} & \cong & \operatorname{spec}(k[t^{\pm}]) \\ \mathfrak{p} & \mapsto & \langle \mathfrak{p} \cap k[t]\rangle_{k[t^{\pm}]} \\ \langle \mathfrak{t} \cap k[t]\rangle_Z + \langle r_n\rangle_Z & \leftarrow\!\shortmid & \mathfrak{t}. \end{array} \tag{94}$$

Thus we can express $\mathfrak{p}$ as $\langle r_n\rangle + \langle \mathfrak{t} \cap k[t]\rangle$, where $\mathfrak{t} = \langle \mathfrak{p} \cap k[t]\rangle_{k[t^{\pm}]} \in \operatorname{spec}(k[t^{\pm}])$.

Fix $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$. Knowing that the image of r_n vanishes in $Z''[u^{\pm}]$ allows us to rewrite $\sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m}$ such that we can factor them:

$$\begin{aligned} \sigma(z) &= -q^{2n}(q^{2n}+1)^{-2}t^2 + tu - u^2 \\ &= -\left(u - (q^{-2n}+1)^{-1}t\right)\left(u - (q^{2n}+1)^{-1}t\right) \\ \sigma^{-n+1}(z) &= -q^{2n}(q^{2n}+1)^{-2}t^2 + q^{-2n}tu - q^{-4n}u^2 \\ &= -q^{-4n}\left(u - (q^{-2n}+1)^{-1}t\right)\left(u - q^{2n}(q^{-2n}+1)^{-1}t\right). \end{aligned} \tag{95}$$

It follows that $\mathfrak{m} = \langle u - \alpha t\rangle$ where either $\alpha = (q^{-2n}+1)^{-1}$ or $\alpha = (q^{2n}+1)^{-1} = q^{2n}(q^{-2n}+1)^{-1}$. The latter case is impossible, for it leads to

$$0 = q^{6n} + q^{4n} - q^{2n} - 1 = (q^{2n}+1)^2(q^{2n}-1).$$

Hence $\mathfrak{m} = \big\langle u - (q^{-2n}+1)^{-1}t\big\rangle$. So gr-spec$(A'') = \{0, J(\big\langle u - (q^{-2n}+1)^{-1}t\big\rangle)\}$. Let $P'' = J(\big\langle u - (q^{-2n}+1)^{-1}t\big\rangle)$. Let P' be the pullback of P'' to A', and let P be the pullback to A. Applying the Chinese Remainder Theorem, we get from Proposition 66 that

$$P''_m = \Big\langle \prod \Big\{ \Big(u - q^{-2j}(q^{-2n}+1)^{-1}t\Big) \mid 0 \le j < n \text{ and } m \le j < n+m \Big\} \Big\rangle_{Z''[u^\pm]}$$

for $m \in \mathbb{Z}$. Contracting these ideals of $Z''[u^\pm]$ to $Z'[u^\pm]$, Lemma 24 tells us that

$$P'_m = \Big\langle \prod \Big\{ \Big(u - q^{-2j}(q^{-2n}+1)^{-1}t\Big) \mid 0 \le j < n \text{ and } m \le j < n+m \Big\} \Big\rangle_{Z'[u^\pm]}$$

for $m \in \mathbb{Z}$. Hence

$$P_m = \langle \pi^n_m \rangle + \langle \mathfrak{p} \rangle = \langle \pi^n_m \rangle + \langle r_n \rangle + \langle \mathfrak{t} \cap k[t] \rangle.$$

for $m \in \mathbb{Z}$. It follows that $P = P(n, \mathfrak{t})$. Thus we have shown that $F_\mathfrak{p} = \{\langle \mathfrak{p} \rangle, P(n, \mathfrak{t} = \langle \mathfrak{p} \cap k[t] \rangle_{k[t^\pm]})\}$.

It is now clear that spec$(\mathcal{A}_u) \subseteq T_2 \cup T_3$. If $\mathfrak{p} \in$ spec(Z) then $\langle \mathfrak{p} \rangle \in F_\mathfrak{p}$, so $T_2 \subseteq$ spec$(\mathcal{A}_u)$. If $n \ge 1$ and $\mathfrak{t} \in$ spec$(k[t^\pm])$, then $P(n, \mathfrak{t}) \in F_\mathfrak{p}$, where $\mathfrak{p} = \langle \mathfrak{p} \cap k[t] \rangle_{k[t^\pm]}$. Thus spec$(\mathcal{A}_u) = T_2 \cup T_3$. ■

3.1.6 The Semiclassical Limit

We now explore the semiclassical limit $\mathcal{A}_1$ of $\mathcal{A}$. Assume throughout this section that k has characteristic zero.

Remark 122: In this development, we defined the k-algebra $\mathcal{A} = \mathcal{A}_q(\mathrm{M}_2)$ in terms of a fixed non-root-of-unity $q \in k$. Another approach would be to start with a $k[\tau^\pm]$-algebra and then specialize to $\tau = q$. To do this, let $\hat{\mathcal{A}}$ be the $k[\tau^\pm]$-algebra generated by u_{ij} for $i, j \in \{1, 2\}$, with the relations coming from a formal replacement of the q in (36) by τ. Then, just as in Proposition 86, we observe that $\hat{\mathcal{A}}$ is a GWA over the polynomial ring $k[\tau^\pm][u_{22}, u_{11}, z]$ and hence that $\hat{\mathcal{A}}$ is a domain (Corollary 6). In particular, $\tau - 1$ is regular in $\hat{\mathcal{A}}$. It is clear from an inspection of the relations for $\hat{\mathcal{A}}$ that $\hat{\mathcal{A}}/\langle \tau - 1\rangle$ is commutative; hence it is a semiclassical limit $\hat{\mathcal{A}}_1$ (that is, it has a Poisson bracket as in Definition 75). And the algebra $\mathcal{A}$ may be recovered by forming the quotient $\hat{\mathcal{A}}/\langle \tau - q\rangle$ and then viewing it as a k-algebra. For convenience of notation and terminology, we shall denote $\hat{\mathcal{A}}_1$ by $\mathcal{A}_1$ and refer to it as "the semiclassical limit of $\mathcal{A}$."

Let us describe $\mathcal{A}_1$. As a k-algebra, it is just the polynomial algebra $k[u_{11}, u_{22}, u_{21}, u_{12}]$. One easily computes the Poisson bracket:

$$\begin{array}{lll} \{u_{11}, u_{22}\} = 0 & \{u_{11}, u_{21}\} = 2u_{21}u_{22} & \{u_{11}, u_{12}\} = -2u_{12}u_{22} \\ \{u_{22}, u_{21}\} = -2u_{21}u_{22} & \{u_{22}, u_{12}\} = 2u_{12}u_{22} & \\ \{u_{21}, u_{12}\} = 2u_{22}(u_{11} - u_{22}). & & \end{array}$$

Since a Poisson bracket is determined by the values it takes on algebra generators, this data determines it. But there is a better way to describe what is going on. Just as we understood $\mathcal{A}$ by viewing it as a GWA, we can understand $\mathcal{A}_1$ by viewing it as a *Poisson* GWA.

Let us apply Theorem 80 to the $2{\times}2$ REA. The $k[\tau^\pm]$-algebra $\hat{\mathcal{A}}$ of Remark 122 is a GWA $k[\tau^\pm][u_{22}, u_{11}, z][x, y; \sigma, z]$, where σ is defined just as in (37), but with the q formally replaced by τ. As with $\mathcal{A}$, a change of variables like (38) gives a simpler expression for $\hat{\mathcal{A}}$ as a GWA,

$$\hat{\mathcal{A}} \cong k[\tau^\pm][u, t, d][x, y; \sigma, z],$$

where $z = d + \tau^{-2}tu - \tau^{-4}u^2$ and σ is defined just as in (40), but with the q formally replaced by τ. Theorem 80 tells us that $\hat{\mathcal{A}}_1$ is

$$k[u,t,d][x,y;\alpha,z]_P, \tag{96}$$

where $k[u,t,d]$ has a trivial Poisson bracket, the element z is $d + tu - u^2$, and α is the derivation of $k[u,t,d]$ given by

$$\begin{aligned}
\alpha(u) &= \frac{\sigma(u) - u}{\tau - 1}\Big|_{\tau=1} = \frac{\tau^2 u - u}{\tau - 1}\Big|_{\tau=1} = (\tau+1)u|_{\tau=1} = 2u \\
\alpha(t) &= \frac{\sigma(t) - t}{\tau - 1}\Big|_{\tau=1} = 0 \\
\alpha(d) &= \frac{\sigma(d) - t}{\tau - 1}\Big|_{\tau=1} = 0.
\end{aligned}$$

As mentioned in Remark 122, we continue to simply refer to $\hat{\mathcal{A}}_1$ as $\mathcal{A}_1$ and as "the semiclassical limit of $\mathcal{A}$."

3.1.6.1 The Poisson Prime Spectrum of $\mathcal{A}_1$

Let A be a Poisson algebra. We will denote by $Z_P(A)$ the Poisson center of A. A *Poisson ideal* P of A is an ideal of A which is also a Lie ideal (i.e. $\{A, P\} \subseteq P$). A proper Poisson ideal P is called *Poisson-prime* if, for all *Poisson* ideals I and J, one has $IJ \subseteq P$ only if $I \subseteq P$ or $J \subseteq P$. A Poisson ideal which is also a prime ideal obviously must be Poisson-prime. When A is noetherian, Poisson-prime ideals are prime (see [19, Lemma 6.2]). In this case we may use the terminology "Poisson prime" (omitting the hyphen) without ambiguity.

The set of Poisson prime ideals of A is denoted by $\operatorname{P.spec}(A)$, and it is given a topology in which closed sets are those of the form

$$V_P(I) := \{P \in \operatorname{P.spec}(A) \mid P \supseteq I\}$$

for ideals I of A. One obtains the same closed sets by considering only *Poisson* ideals for I, since replacing I by the Poisson ideal it generates yields the same $V_P(I)$.

The following facts are standard:

Proposition 123: *The quotient of a Poisson algebra A by a Poisson ideal I is again a Poisson algebra, with a well-defined induced Poisson bracket. Pullback of ideals provides a bijective correspondence between Poisson ideals of A/I and the Poisson ideals of A that contain I, and it provides a homeomorphism between* $\operatorname{P.spec}(A/I)$ *and the subset of* $\operatorname{P.spec}(A)$ *consisting of Poisson primes that contain I.*

Proposition 124: *The localization of a Poisson algebra A at a multiplicative set $\mathcal{S}$ is again a Poisson algebra with an induced Poisson bracket. Pullback of prime ideals provides a homeomorphism between* $\operatorname{P.spec}(A\mathcal{S}^{-1})$ *and the subset of* $\operatorname{P.spec}(A)$ *consisting of Poisson primes that are disjoint from $\mathcal{S}$.*

If α is a Poisson derivation of a Poisson algebra R and J is a Poisson ideal of R such that $\alpha(J) \subseteq J$, then α induces a derivation $\hat{\alpha}$ of the quotient R/J (Proposition 149), and it is clear that $\hat{\alpha}$ is a *Poisson* derivation. We have an an obvious analogue to Proposition 12:

Proposition 125: *Let $W = R[x,y;\alpha,z]_P$ be a PGWA, with $J \triangleleft R$ a Poisson ideal such that $\alpha(J) \subseteq J$. Let $I \triangleleft W$ be generated by J. Then I is a Poisson ideal and there is a canonical isomorphism*

$$W/I \cong (R/J)[x,y;\hat{\alpha}, z + J]_P, \tag{97}$$

where $\hat{\alpha}$ is the Poisson derivation of R/J induced by α.

Proof: Since $\alpha(J) \subseteq J$, we have $\{v_m, J\} \subseteq I$ for $m \in \mathbb{Z}$. Therefore $\{W, J\} \subseteq I$, and it follows easily that I is a Poisson ideal of W. The rest of the Proposition is routine. ■

We also have an analogue to Propositions 2 and 5. For this we refer to the notion of a Poisson polynomial ring $R[x;\alpha]_P$ developed in [36], and the notion of Poisson Laurent polynomial ring $R[x^{\pm};\alpha]_P$ that arises by localization. Our convention differs from [36] in that $R[x;\alpha]_P$ stands for what is called $R[x;-\alpha]_P$ in [36]. We will later also use the notation $R(x;\alpha)_P$ to refer to the "Poisson function field" (this is again just constructed by applying localization to Poisson polynomial rings).

Proposition 126: *Let $R[x,y;\alpha,z]_P$ be a Poisson GWA. The homomorphisms $\phi, \phi' : R[x,y;\alpha,z]_P \to R[x^{\pm};\alpha]_P$ of Proposition 2 are Poisson algebra homomorphisms. They are injective when z is regular and they are isomorphisms when z is a unit.*

Finally, we put down the obvious Poisson versions of Propositions 21 and 22:

Proposition 127: *Let $W = R[x,y;\alpha,z]_P$ be a Poisson GWA with z regular. Then the localization of W at the multiplicative set $\mathcal{S} := \{1, x, x^2, \ldots\}$ is given by the Poisson algebra homomorphism $\phi : W \to R[x^{\pm};\alpha]_P$ of Proposition 126.*

Proposition 128: *Let $W = R[x,y;\alpha,z]_P$ be a Poisson GWA and let $\mathcal{S} \subseteq R$ be a multplicative set. Let $\hat{\alpha}$ be the derivation of $R\mathcal{S}^{-1}$ induced by α, and let $\hat{z}$ be the image of z in $R\mathcal{S}^{-1}$. The localization of W at $\mathcal{S}$ is given by the map $\phi : W \to R\mathcal{S}^{-1}[x,y;\hat{\alpha},\hat{z}]_P$ of Proposition 22, and this map is a Poisson algebra homomorphism.*

It often happens that the prime spectrum of a quantized algebra lines up with the Poisson prime spectrum of its semiclassical limit. "Lines up" here means only that there is some bijection that matches up ideals that are expressed using the same notation; it is difficult to find general theoretical grounds for such bijections. The pattern, however, has been that a correspondence exists when the prime ideals of a quantized algebra are all completely prime. Since Poisson prime ideals are in particular prime ideals of a commutative algebra, one would not expect them to ever accurately resemble non-completely-prime primes of a noncommutative algebra. In the case of our REA, the primes in T_{3n} for $n > 1$ are not completely prime, and indeed we will see that they have no analogue in $\mathrm{P.spec}(\mathcal{A})$. As for other primes of the REA, they will have analogous Poisson primes.

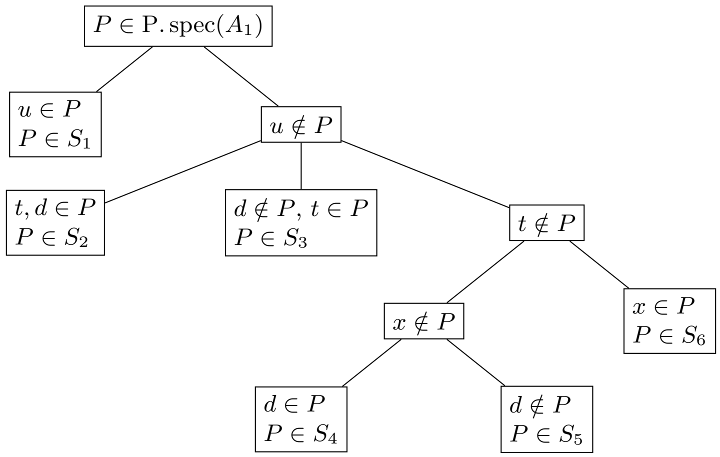
For the rest of this section, fix the notation $R = k[u,t,d]$. Define the following subsets of $\mathrm{P.spec}(\mathcal{A})$:

$$\begin{aligned}
T_1^P &= \{P \in \mathrm{P.spec}(\mathcal{A}_1) \mid u \in P\},\\
T_2^P &= \{P \in \mathrm{P.spec}(\mathcal{A}_1) \mid P = \langle P \cap k[t,d] \rangle\}, \text{ and}\\
T_{3,1}^P &= \{P \in \mathrm{P.spec}(\mathcal{A}_1) \mid u,t \notin P,\ x \in P\}.
\end{aligned}$$

Theorem 129: *Assume that* $\mathrm{char}(k) = 0$. *The Poisson spectrum of the semiclassical limit $\mathcal{A}_1$ of $\mathcal{A}$ is, as a set, the disjoint union of T_1^P, T_2^P, and $T_{3,1}^P$. Each of these subsets is homeomorphic to a commutative prime spectrum as follows:*

- $\mathrm{spec}(k[u_{11},u_{12},u_{21}]) \approx T_1^P$ *via* $\mathfrak{p} \mapsto \langle u \rangle + \langle \mathfrak{p} \rangle$.
- $\mathrm{spec}(k[t,d]) \approx T_2^P$ *via* $\mathfrak{p} \mapsto \langle \mathfrak{p} \rangle$.
- $\mathrm{spec}(k[t^{\pm}]) \approx T_{3,1}^P$ *via* $\mathfrak{p} \mapsto \langle x, y, t-2u \rangle + \langle \mathfrak{p} \cap k[t] \rangle$.

Proof: Consider the partition of $\mathrm{P.spec}(\mathcal{A}_1)$ into subsets $S_1, \ldots, S_6$ given by the following tree, in which branches represent mutually exclusive possibilities:



It is clear that $S_1 = T_1^P$ and $S_6 = T_{3,1}^P$. To show that $\{T_1^P, T_2^P, T_{3,1}^P\}$ is a partition of $\mathrm{P.spec}(\mathcal{A}_1)$. we can show that

$$T_2^P = S_2 \cup S_3 \cup S_4 \cup S_5. \tag{98}$$

For the inclusion $\subseteq$, consider a $P = \langle \mathfrak{p} \rangle_{\mathcal{A}_1} \in T_2^P$, where $\mathfrak{p} \in \mathrm{spec}(k[t,d])$. Note that $P = \bigoplus_{m \in \mathbb{Z}} \langle \mathfrak{p} \rangle_R v_m$ (the inclusion $\subseteq$ is clear, and $\supseteq$ follows from the fact that the right hand side is an ideal). Now it's clear that $u \notin P$, since $u \in P \Rightarrow u \in P_0 = \langle \mathfrak{p} \rangle_R$, and it's clear that $x \notin P$, since $P_1 = \langle \mathfrak{p} \rangle_R$ is proper. Hence P cannot be in S_1 or S_6, proving $\subseteq$ in (98). For the reverse inclusion, we will show individually that each of $S_2, \ldots, S_5$ is contained in T_2^P.

$S_2 \subseteq T_2^P$: Let $(\mathcal{A}_1)_u$ denote the localization of $\mathcal{A}_1$ at u. By Propositions 123 and 124, elements of S_2 correspond to Poisson primes of $(\mathcal{A}_1)_u/\langle t,d \rangle$. By Propositions 128 and 125, the algebra $(\mathcal{A}_1)_u/\langle t,d \rangle$ is isomorphic to $k[u^\pm][x,y;\alpha, z = -u^2]_P$. This is in turn isomorphic to $k[u^\pm, x^\pm]$ by Proposition 126. According to [19, 9.6(b)], contraction and extension give inverse homeomorphisms between $\mathrm{P.spec}(k[u^\pm, x^\pm])$ and $\mathrm{spec}(Z_P(k[u^\pm, x^\pm]))$. It is also observed in [19, 9.6(b)] that since the Poisson bracket respects the standard $\mathbb{Z}^2$-grading of $k[u^\pm, x^\pm]$, the Poisson center $Z_P(k[u^\pm, x^\pm])$ is spanned by the monomials that it contains. In order for a monomial $u^i x^j$ to be Poisson central, one needs $0 = \{u^i x^j, u\} = j x^{j-1} u^i \{x, u\} = 2j x^j u^{i+1}$ and $0 = \{u^i x^j, x\} = i u^{i-1} x^j \{u, x\} = -2i u^i x^{j+1}$, which requires $i = j = 0$ (using the characteristic zero hypothesis). Thus $Z_P(k[u^\pm, x^\pm]) = k$, and $k[u^\pm, x^\pm]$ has only one Poisson prime, $\langle 0 \rangle$, which pulls back to $\langle t, d \rangle \triangleleft \mathcal{A}_1$. Hence $S_2 = \{\langle t, d \rangle\} \subseteq T_2^P$.

$S_3 \subseteq T_2^P$: Let $(\mathcal{A}_1)_{ud}$ denote the localization of $\mathcal{A}_1$ at u and d. By Propositions 123 and 124, elements of S_3 correspond to Poisson primes of $(\mathcal{A}_1)_{ud}/\langle t \rangle$. By Propositions 128 and 125, the algebra $(\mathcal{A}_1)_{ud}/\langle t \rangle$ is isomorphic to $k[u^\pm, d^\pm][x,y;\alpha, z = d - u^2]_P$. A Poisson prime ideal of $k[u^\pm, d^\pm][x,y;\alpha, z = d - u^2]_P$ cannot contain x, because it would then contain the unit $\{x,y\} = \alpha(d - u^2) = -4u^2$. By Proposition 127, elements of S_3 therefore correspond to Poisson primes of $k[u^\pm, d^\pm, x^\pm]$. Again using [19, 9.6(b)], contraction and extension give inverse homeomorphisms between $\mathrm{P.spec}(k[u^\pm, d^\pm, x^\pm])$ and $\mathrm{spec}(Z_P(k[u^\pm, d^\pm, x^\pm]))$. And again $Z_P(k[u^\pm, d^\pm, x^\pm])$ is spanned by the monomials that it contains. In order for a monomial $u^i d^j x^l$ to be Poisson central, one needs $0 = \{u^i d^j x^l, u\} = 2l u^{i+1} d^j x^l$ and $0 = \{u^i d^j x^l, x\} = -2i u^i d^j x^{l+1}$, which require $i = l = 0$. Thus $Z_P(k[u^\pm, d^\pm, x^\pm]) = k[d^\pm]$. Any prime $\mathfrak{p} \in \mathrm{spec}(k[d^\pm])$ corresponds, via our chain of correspondences, to the Poisson prime $\langle \mathfrak{p} \cap k[d] \rangle + \langle t \rangle \triangleleft \mathcal{A}_1$. Since this is in T_2^P, we conclude that $S_3 \subseteq T_2^P$.

$S_4 \subseteq T_2^P$: Let $(\mathcal{A}_1)_{utx}$ denote the localization of $\mathcal{A}_1$ at u, t, and x. By Propositions 123 and 124, elements of S_4 correspond to Poisson primes of $(\mathcal{A}_1)_{utx}/\langle d \rangle$. By Propositions 128, 127, and 125, the algebra $(\mathcal{A}_1)_{utx}/\langle d \rangle$ is isomorphic to $k[u^\pm, t^\pm, x^\pm]$. The Poisson algebra structure of $k[u^\pm, t^\pm, x^\pm]$ is identical to that of $k[u^\pm, d^\pm, x^\pm]$ in the preceding paragraph, if one replaces the t by d. Therefore we have $\mathrm{P.spec}(k[u^\pm, t^\pm, x^\pm]) \approx \mathrm{spec}(k[t^\pm])$ by the preceding paragraph. Any prime $\mathfrak{p} \in \mathrm{spec}(k[t^\pm])$ corresponds, via our chain of correspondences, to the Poisson prime $\langle \mathfrak{p} \cap k[t] \rangle + \langle d \rangle \triangleleft \mathcal{A}_1$. Since this is in

T_2^P, we conclude that $S_4 \subseteq T_2^P$.

$S_5 \subseteq T_2^P$: By Propositions 123 and 124, elements of S_5 correspond to Poisson primes of $(\mathcal{A}_1)_{utdx}$, the localization of $\mathcal{A}_1$ at u, t, d, and x. By Propositions 128, 127, and 125, the algebra $(\mathcal{A}_1)_{utdx}$ is isomorphic to $k[u^\pm, t^\pm, d^\pm, x^\pm]$. Again using [19, 9.6(b)], contraction and extension give inverse homeomorphisms between $\mathrm{P.spec}(k[u^\pm, t^\pm, d^\pm, x^\pm])$ and $\mathrm{spec}(Z_P(k[u^\pm, t^\pm, d^\pm, x^\pm]))$. And again $Z_P(k[u^\pm, t^\pm, d^\pm, x^\pm])$ is spanned by the monomials that it contains. In order for a monomial $u^i t^j d^l x^m$ to be Poisson central, one needs $0 = \{u^i t^j d^l x^m, u\} = 2mu^{i+1} t^j d^l x^m$ and $0 = \{u^i t^j d^l x^m, x\} = -2iu^i t^j d^l x^{m+1}$, which require $i = m = 0$. Thus $Z_P(k[u^\pm, t^\pm, d^\pm, x^\pm]) = k[t^\pm, d^\pm]$. Any prime $\mathfrak{p} \in \mathrm{spec}(k[t^\pm, d^\pm])$ corresponds, via our chain of correspondences, to the Poisson prime $\langle \mathfrak{p} \cap k[t,d] \rangle \triangleleft \mathcal{A}_1$. Since this is in T_2^P, we conclude that $S_5 \subseteq T_2^P$.

We now have the equality (98), and thus we've shown that $\mathrm{P.spec}(\mathcal{A}_1) = T_1^P \sqcup T_2^P \sqcup T_{3,1}^P$ as a set. It remains to address the homeomorphisms. The homeomorphism $T_1^P \approx \mathrm{P.spec}(k[u_{11}, u_{12}, u_{21}]) = \mathrm{spec}(k[u_{11}, u_{12}, u_{21}])$ comes from the fact that $\mathcal{A}_1/\langle u \rangle$ is $k[u_{11}, u_{12}, u_{21}]$ with a trivial Poisson bracket.

Consider the partition of $\mathrm{spec}(k[t,d])$ into the subspaces

$$\begin{array}{ll} S_2' := \{\langle t, d \rangle\} & S_3' := \{\mathfrak{p} \in \mathrm{spec}(k[t,d]) \mid t \in \mathfrak{p},\ d \notin \mathfrak{p}\} \\ S_4' := \{\mathfrak{p} \in \mathrm{spec}(k[t,d]) \mid t \notin \mathfrak{p},\ d \in \mathfrak{p}\} & S_5' := \{\mathfrak{p} \in \mathrm{spec}(k[t,d]) \mid t \notin \mathfrak{p},\ d \notin \mathfrak{p}\}, \end{array}$$

which are respectively homeomorphic to $\mathrm{spec}(k)$, $\mathrm{spec}(k[d^\pm])$, $\mathrm{spec}(k[t^\pm])$, and $\mathrm{spec}(k[t^\pm, d^\pm])$. For each $i \in \{2,3,4,5\}$, we showed above (in the paragraph labeled "$S_i \subseteq T_2^P$") that the mapping $\mathfrak{p} \mapsto \langle \mathfrak{p} \rangle \triangleleft \mathcal{A}_1$ sends S_i' bijectively to S_i. It follows, given (98), that the mapping $\mathfrak{p} \mapsto \langle \mathfrak{p} \rangle \triangleleft \mathcal{A}_1$ gives a bijection $\phi : \mathrm{spec}(k[t,d]) \to T_2^P$. For ϕ to be a homeomorphism, it suffices, by [19, Lemma 9.4a], for ϕ and ϕ^{-1} to preserve inclusions. It is obvious that ϕ preserves inclusions. To make it clear that ϕ^{-1} preserves inclusions, note that

$$\langle \mathfrak{p} \rangle_{\mathcal{A}_1} = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{n \geq 0} \mathfrak{p} u^n v_m$$

for $\mathfrak{p} \in \mathrm{spec}(k[t,d])$, so $\langle \mathfrak{p} \rangle \cap k[t,d] = \mathfrak{p}$. Hence ϕ is a homeomorphism.

Now for the final homeomorphism. Consider any $P \in T_{3,1}^P$. Since $x \in P$, we have $P \ni \{x, y\} = \alpha(z) = 2tu - 4u^2$. Since $u \notin P$, it follows that $t - 2u \in P$. From this it follows that $P \ni \{y, t - 2u\} = 4uy$, and again since $u \notin P$ we have $y \in P$. Similarly, from $P \ni \{t - 2u, x\} = 4ux$ we conclude that $x \in P$. Therefore any element of $T_{3,1}^P$ contains the Poisson ideal $\langle x, y, t - 2u \rangle$. Let $(\mathcal{A}_1)_{ut}$ denote the localization of $\mathcal{A}_1$ at u and t. By Propositions 123 and 124, the subspace $T_{3,1}^P$ of $\mathrm{P.spec}(\mathcal{A}_1)$ is homeomorphic to $\mathrm{P.spec}((\mathcal{A}_1)_{ut}/\langle x, y, t-2u \rangle)$. Observe that $(\mathcal{A}_1)_{ut}/\langle x, y, t-2u \rangle$ is isomorphic to $k[t^\pm]$ with a trivial Poisson bracket. Hence we have $\mathrm{P.spec}((\mathcal{A}_1)_{ut}/\langle x, y, t-2u \rangle) \approx \mathrm{spec}(k[t^\pm])$. Following a $\mathfrak{p} \in \mathrm{spec}(k[t^\pm])$ along our chain of homeomorphisms $T_{3,1}^P \approx \mathrm{P.spec}((\mathcal{A}_1)_{ut}/\langle x, y, t-2u \rangle) \approx \mathrm{spec}(k[t^\pm])$, we see that $\mathfrak{p}$ corresponds to $\langle \mathfrak{p} \cap k[t] \rangle + \langle x, y, t - 2u \rangle \in T_{3,1}^P$. ■

There is a Poisson analogue to the Dixmier-Moeglin equivalence, in terms of the following notions. A *Poisson primitive* ideal of a Poisson algebra is defined to be any ideal that arises as the largest Poisson ideal contained in a maximal ideal. Poisson primitives can be shown to be Poisson prime (see [19, Lemma 6.2]). A *Poisson rational* ideal P of a Poisson algebra A is a Poisson prime ideal such that $Z_P(\mathrm{Fract}(A)/P)$ is algebraic over k. A Poisson algebra A satisfies the *Poisson Dixmier-Moeglin equivalence* when these conditions are equivalent for any $P \in \mathrm{P.spec}(A)$ and are equivalent to $\{P\}$ being locally closed.

Theorem 130: *Assume that* $\mathrm{char}(k) = 0$. *The semiclassical limit* $\mathcal{A}_1$ *of* $\mathcal{A}$ *satisfies the Poisson Dixmier-Moeglin equivalence and its Poisson primitive ideals are as follows:*

- *The Poisson primitive ideals in* T_1^P *are* $\langle u \rangle + \langle \mathfrak{p} \rangle$ *for* $\mathfrak{p} \in \max\mathrm{spec}\, k[u_{11}, u_{12}, u_{21}]$.
- *The Poisson primitive ideals in* T_2^P *are* $\langle \mathfrak{p} \rangle$ *for* $\mathfrak{p} \in \max\mathrm{spec}\, k[t,d]$.
- *The Poisson primitive ideals in* $T_{3,1}^P$ *are* $\langle \mathfrak{p} \cap k[t] \rangle + \langle x, y, t - 2u \rangle$ *for* $\mathfrak{p} \in \max\mathrm{spec}\, k[t^\pm]$.

Proof: Since $\mathcal{A}_1$ is affine, we have by [18, Proposition 1.2] that the following implications hold for all prime ideals of $\mathcal{A}$:

$$\text{locally closed} \quad \Longrightarrow \quad \text{Poisson primitive} \quad \Longrightarrow \quad \text{Poisson rational.}$$

To establish the Poisson Dixmier-Moeglin equivalence for $\mathcal{A}_1$, it remains to close the loop and show that Poisson rational primes are locally closed. We shall deal separately with the three different types of Poisson primes identified in Theorem 129.

T_1^P**:** Suppose that $P \in T_1^P$, say $P = \langle \mathfrak{p} \rangle + \langle u \rangle$, with $\mathfrak{p} \in \operatorname{spec}(k[u_{11}, u_{12}, u_{21}])$. Since $Z_P(\mathcal{A}_1/P) = \mathcal{A}_1/P \cong k[u_{11}, u_{12}, u_{21}]/\mathfrak{p}$, the ideal P is Poisson rational if and only if $\mathfrak{p}$ is maximal. In this case, P would be maximal and therefore locally closed in $\mathrm{P.spec}(\mathcal{A}_1)$.

T_2^P**:** Suppose that $P \in T_2^P$, say $P = \langle \mathfrak{p} \rangle$, with $\mathfrak{p} \in \operatorname{spec}(k[t, d])$. By Proposition 125,

$$\mathcal{A}_1/P \cong (k[u, t, d]/\langle \mathfrak{p} \rangle)[x, y; \alpha, z = d + tu - u^2]_P.$$

Since z is regular in $k[u, t, d]/\langle \mathfrak{p} \rangle$, we have by Proposition 126 an embedding

$$\mathcal{A}_1/P \hookrightarrow (k[u, t, d]/\langle \mathfrak{p} \rangle)[x^{\pm}; \alpha]_P = (k[t, d]/\mathfrak{p})[u][x^{\pm}; \alpha]_P$$

of Poisson algebras. There is a corresponding embedding of fraction fields:

$$\mathrm{Fract}(\mathcal{A}_1/P) \hookrightarrow \mathrm{Fract}(k[t, d]/\mathfrak{p})(u)(x; \alpha)_P.$$

Let $L = \mathrm{Fract}(k[t, d]/\mathfrak{p})$. In order for f to be an element of $Z_P(L(u)(x; \alpha)_P)$, it is enough for $\{f, -\}$ to vanish on L, on u, and on x. Since $L \cup \{u, x\} \subseteq \mathrm{Fract}(\mathcal{A}_1/P)$, it follows that

$$Z_P(\mathrm{Fract}(\mathcal{A}_1/P)) = Z_P(L(u)(x; \alpha)_P) \cap \mathrm{Fract}(\mathcal{A}_1/P). \tag{99}$$

We can compute $Z_P(L(u)(x; \alpha)_P)$:

Claim: $Z_P(L(u)(x; \alpha)_P) = L$
Proof: Consider any nonzero element $\frac{f}{g} \in Z_P(L(u)(x; \alpha)_P)$, where $f, g \in L[u, x]$ are taken to be nonzero and coprime. Since $\left\{ \frac{f}{g}, - \right\} = 0$, we have $f\{g, -\} = g\{f, -\}$. Applying this identity to u and cancelling $2ux$ gives

$$f \frac{\partial g}{\partial x} = g \frac{\partial f}{\partial x},$$

and applying it to x and cancelling $-2ux$ gives

$$f \frac{\partial g}{\partial u} = g \frac{\partial f}{\partial u}.$$

Since we assumed f and g to be coprime, this implies that $f \mid \frac{\partial f}{\partial x}$, that $g \mid \frac{\partial g}{\partial x}$, that $f \mid \frac{\partial f}{\partial u}$, and that $g \mid \frac{\partial g}{\partial u}$. Considering degrees of polynomials, this can only happen if $0 = \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \frac{\partial f}{\partial u} = \frac{\partial g}{\partial u}$. Thus $f, g \in L$.

Using (99) we now have $Z_P(\mathrm{Fract}(\mathcal{A}_1/P)) = L$. Note that L is algebraic over k if and only if $\mathfrak{p}$ is maximal. Hence P is Poisson rational if and only if $\mathfrak{p}$ is maximal. Now assume that P is Poisson rational. By Theorem 129, any Poisson prime of $\mathcal{A}_1$ that doesn't contain u or x must be in $T_2^P \approx \operatorname{spec}(k[t, d])$. This makes T_2^P an open subset of $\mathrm{P.spec}(\mathcal{A}_1)$. Since $\mathfrak{p}$ is maximal, it is a closed point of $\operatorname{spec}(k[t, d])$; it follows that P is a closed point of T_2^P. Therefore P is locally closed in $\mathrm{P.spec}(\mathcal{A}_1)$.

$T_{3,1}^P$**:** Suppose that $P \in T_{3,1}^P$, say $P = \langle \mathfrak{p} \cap k[t] \rangle + \langle x, y, t - 2u \rangle$, with $\mathfrak{p} \in \operatorname{spec}(k[t^{\pm}])$. Viewing $\mathcal{A}_1$ as $k[u_{11}, u_{22}, u_{21}, u_{12}] = k[u, t, x, y]$ (recall that $t = u_{11} + u_{22}$), it is easy to see that

$$k[t] \hookrightarrow \mathcal{A}_1 \xrightarrow{quo} \mathcal{A}_1/\langle x, y, t - 2u \rangle$$

is an isomorphism of algebras which have a trivial Poisson bracket. So $\mathcal{A}_1/P \cong k[t]/(\mathfrak{p} \cap k[t])$. Hence $Z_P(\mathrm{Fract}(\mathcal{A}_1/P)) = \mathrm{Fract}(\mathcal{A}_1/P)$ is algebraic over k if and only if $\mathfrak{p} \cap k[t] \triangleleft k[t]$ is maximal. This occurs if and only if $\mathfrak{p} \triangleleft k[t^{\pm}]$ is maximal. Thus P is Poisson rational if and only if $\mathfrak{p}$ is maximal. Now assume that P is Poisson rational, and let us use the description of Poisson primes in Theorem 129 to show that P is locally closed. No element of T_2^P contains P, for elements of T_2^P cannot contain x. Since $\mathfrak{p}$ is maximal, we have that P is maximal inside $T_{3,1}^P$. Therefore $\{P\} = V_P(P) \setminus T_1^P = V_P(P) \setminus V_P(u)$ is locally closed.

We have now shown that all Poisson rational prime ideals of $\mathcal{A}_1$ are locally closed, and we conclude that $\mathcal{A}_1$ satisfies the Poisson Dixmier-Moeglin equivalence. Further, we have pinpointed which Poisson primes are Poisson rational in each of T_1^P, T_2^P, and $T_{3,1}^P$. Putting this information together and applying the Poisson Dixmier-Moeglin equivalence, we conclude that the Poisson primitive ideals of $\mathcal{A}_1$ are as stated in the theorem. ■

3.2 Weyl Algebra

In this section we treat the classical and quantum Weyl algebras.

The classical Weyl algebra $A_1(k)$ is defined to be the k-algebra generated by x and y with the relation $xy - yx = 1$. It is well known that $A_1(k)$ is a simple ring when k has characteristic 0. This can be shown using the theory of skew polynomial rings, for example (see [21, Chapter 2]). The Weyl algebra is a GWA $k[z][x, y; \sigma : z \mapsto z + 1, z]$, and we can also apply GWA theory to see that it is simple:

Theorem 131: *Assume k has characteristic 0. Then $A_1(k)$ is a simple ring.*

Proof: Let $A := A_1(k) = k[z][x, y; \sigma : z \mapsto z + 1, z]$. All ideals of A are homogeneous because the commutator $[\cdot, z]$ is a linear operator that has the graded components of A as eigenspaces with distinct eigenvalues (see Proposition 81). By Proposition 83, Corollary 71 applies to A. Clearly $\mathscr{M}'_{\mathrm{II}} = \emptyset$, since the $\sigma^i(z)$ are pairwise coprime. Thus $\mathrm{spec}(A) = \text{gr-spec}(A) = \{0\}$, and it follows that A is simple. ■

Now consider the quantized Weyl Algebra $A_1(k, q)$, where $q \in k^\times$ is not a root of unity. This is defined to be the k-algebra generated by x and y with the relation $xy - qyx = 1$. The prime spectrum of $A_1(k, q)$ was studied in [17]. Here we use GWA theory to determine the prime spectrum, reproducing [17, Theorem 8.4b].

The quantized Weyl algebra is a GWA $k[z][x, y; \sigma : z \mapsto qz + 1, z]$. By a change of variables to $u = z - \frac{1}{1-q}$, we can express $A_1(k, q)$ as $k[u][x, y; \sigma : u \mapsto qu, z = u + \frac{1}{1-q}]$. Define an algebra homomorphism $\eta : A \to k[x^{\pm}]$ by $u \mapsto 0$, $x \mapsto x$, and $y \mapsto \frac{1}{1-q}x^{-1}$.

Theorem 132: *Assume that $q \in k^\times$ is not a root of unity. As a set,*

$$\mathrm{spec}(A_1(k, q)) = \{\eta^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{spec}(k[x^{\pm}])\} \cup \{0\}.$$

Proof: Consider any prime ideal P of $A := A_1(k, q)$. If P contains u, then P corresponds to a prime ideal of $A/\langle u \rangle \cong k[x, y; \sigma = \mathrm{id}, z = \frac{1}{1-q}] \cong k[x^{\pm}]$ (here we have used Propositions 12 and 5). If P corresponds to $\mathfrak{p} \in \mathrm{spec}(k[x^{\pm}])$, then $P = \eta^{-1}(\mathfrak{p}) \in \{\eta^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{spec}(k[x^{\pm}])\} \cup \{0\}$.

Suppose now that P does not contain u. Then P corresponds to some prime ideal P' of the localization $A_u := k[u^{\pm}][x, y; \sigma, z]$. Since q is not a root of unity, conjugation by u is a linear operator on A_u that has the graded components of A_u as eigenspaces with distinct eigenvalues. It follows that P' is homogeneous (Proposition 81). By Proposition 82, every maximal ideal of $k[u^{\pm}]$ has infinite σ-orbit. Hence Corollary

71 applies and we have $P' \in \{0\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \max\operatorname{spec} k[u^{\pm}]\}$. Using the fact that q is not a root of unity, it is easy to calculate that $\left\langle \sigma^{n'}(z), \sigma^{-n+1}(z) \right\rangle_{k[u^{\pm}]} = \langle 1 \rangle$ for any integers $n, n' \geq 1$. It follows that $\mathscr{M}'_{\mathrm{II}} = \emptyset$, so $P' = 0$ and therefore $P = 0$.

Thus we have proven that $\operatorname{spec}(A) \subseteq \{P(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{spec}(k[x^{\pm}])\} \cup \{0\}$. It is clear that $\{\eta^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{spec}(k[x^{\pm}])\} \subseteq \operatorname{spec}(A)$. Finally, $0 \in \operatorname{spec}(A)$ because A is a domain (Corollary 6). ■

3.3 Weyl-Like Algebras with General z

Consider a GWA of the form GWA $A(z) := k[H][x, y; \sigma : H \mapsto H - 1, z]$ with arbitrary nonzero $z \in k[H] \setminus \{0\}$. This is the main object of study in [4], but the prime spectrum was not determined. Assuming that k has characteristic zero and that z splits into linear factors, we will write down the prime spectrum of $A(z)$.

It is more convenient to work with a splitting of $\sigma(z)$ into linear factors– say $\sigma(z)$ is a unit multiple of $(H - \lambda_1)^{\alpha_1} \cdots (H - \lambda_d)^{\alpha_d}$, where the λ_i are distinct elements of k and where the α_i are in $\mathbb{Z}_{\geq 1}$. Let

$$\mathscr{I} = \{i \mid 1 \leq i \leq d \text{ and } \lambda_i + n \in \{\lambda_1, \ldots, \lambda_d\} \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

For $i \in \mathscr{I}$ and $m \in \mathbb{Z}$, define

$$\pi_m(i) = \prod \{H - (\lambda_i + \ell) \mid 0 \leq \ell < n(i) \text{ and } m \leq \ell < n(i) + m\} \in k[H],$$

where $n(i) := \min\{n > 0 \mid \lambda_i + n \in \{\lambda_1, \ldots, \lambda_d\}\}$. Define also $P(i) = \langle \pi_m(i) v_m \mid m \in \mathbb{Z} \rangle \triangleleft A(z)$ for $i \in \mathscr{I}$.

Theorem 133: *Assume that k has characteristic 0 and that $z \in k[H]$ is nonzero and splits into linear factors as described above. Then as a set,* $\operatorname{spec}(A(z)) = \{P(i) \mid i \in \mathscr{I}\} \cup \{0\}$.

Proof: The linear operator $[\cdot, H]$ on $A(z)$ has the graded components as eigenspaces with distinct eigenvalues; hence all ideals of $A(z)$ are homogeneous (Proposition 81). By Proposition 83, all maximal ideals of $k[H]$ have infinite σ-orbit. Hence Corollary 71 applies and we have $\operatorname{spec}(A(z)) = \{0\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}\}$. Fix $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$, and let $n = n(\mathfrak{m})$. Since $\sigma(z) \in \mathfrak{m}$, we have $\mathfrak{m} = \langle H - \lambda_i \rangle$ for some i with $1 \leq i \leq d$. Since $\sigma^{-n}(\sigma(z)) \in \mathfrak{m} = \langle H - \lambda_i \rangle$, we have $\lambda_i = \lambda_j - n$ for some j with $1 \leq j \leq d$. Hence $i \in \mathscr{I}$. We also have

$$\begin{aligned} J(\mathfrak{m})_m &= \bigcap \{\sigma^{\ell}(\mathfrak{m}) \mid 0 \leq \ell < n \text{ and } m \leq \ell < n + m\} \\ &= \bigcap \{H - (\lambda_i + \ell) \mid 0 \leq \ell < n \text{ and } m \leq \ell < n + m\} \\ &= \left\langle \prod \{H - (\lambda_i + \ell) \mid 0 \leq \ell < n \text{ and } m \leq \ell < n + m\} \right\rangle \end{aligned}$$

for $m \in \mathbb{Z}$, by Proposition 66 and the Chinese remainder theorem. Since $n = n(\mathfrak{m}) = \min\{\overline{n} > 0 \mid \lambda_i + \overline{n} \in \{\lambda_1, \ldots, \lambda_d\}\} = n(i)$, the final line is $\langle \pi_m(i) \rangle$. Thus $J(\mathfrak{m}) = P(i)$.

Conversely to the above, we clearly have for any $i \in \mathscr{I}$ that $\langle H - \lambda_i \rangle \in \mathscr{M}'_{\mathrm{II}}$. Thus $\mathscr{M}'_{\mathrm{II}} = \{\langle H - \lambda_i \rangle \mid i \in \mathscr{I}\}$, and we have proven that $\operatorname{spec}(A(z)) = \{P(i) \mid i \in \mathscr{I}\} \cup \{0\}$. ■

Let us also look at a variation on $A(z)$ that is analogous to the *quantized* Weyl algebra.

Consider a GWA of the form $A(q, z) := k[h^{\pm}][x, y; \sigma : h \mapsto qh, z]$, where $q \in k^{\times}$ and $z \in k[h^{\pm}] \setminus \{0\}$. This algebra makes an appearance as $A(a(h), q)$ in [12], where its semiclassical limit is explored. The prime spectrum has a very similar description to that of $A(z)$– in fact the result and proof are almost identical modulo notation!

Let $D = k[h^{\pm}]$. Assume that $q \in k^{\times}$ is not a root of unity. Assume z is any nonzero element of D that splits into linear factors. It is more convenient to work with a splitting of $\sigma(z)$ into linear factors– say $\sigma(z)$ is a unit multiple of $(h-\lambda_1)^{\alpha_1}\cdots(h-\lambda_d)^{\alpha_d}$, where the λ_i are distinct elements of k and where the α_i are in $\mathbb{Z}_{\geq 1}$. Let

$$\mathscr{I} = \{i \mid 1 \leq i \leq d \text{ and } q^{-n}\lambda_i \in \{\lambda_1, \ldots, \lambda_d\} \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

For $i \in \mathscr{I}$ and $m \in \mathbb{Z}$, define

$$\pi_m(i) = \prod\{h - q^{-\ell}\lambda_i \mid 0 \leq \ell < n(i) \text{ and } m \leq \ell < n(i)+m\},$$

where $n(i) := \min\{n > 0 \mid q^{-n}\lambda_i \in \{\lambda_1, \ldots, \lambda_d\}\}$. Define $P(i) = \langle \pi_m(i)v_m \mid m \in \mathbb{Z}\rangle$ for $i \in \mathscr{I}$.

Theorem 134: *Assume that q is not a root of unity and that $z \in k[h^{\pm}]$ is nonzero and splits into linear factors as described above. Then as a set,* $\operatorname{spec}(A(q,z)) = \{P(i) \mid i \in \mathscr{I}\} \cup \{0\}$.

Proof: The linear operator on $A(q,z)$ given by $a \mapsto h^{-1}ah$ has as eigenspaces the graded components of $A(q,z)$ (with distinct eigenvalues because q is not a root of unity); hence all ideals of $A(q,z)$ are homogeneous (Proposition 81). By Proposition 82, all maximal ideals of D have infinite σ-orbit. Hence Corollary 71 applies and we have $\operatorname{spec}(A(z)) = \{0\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}\}$. Fix $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$, and let $n = n(\mathfrak{m})$. Since $\sigma(z) \in \mathfrak{m}$, we have $\mathfrak{m} = \langle h - \lambda_i\rangle$ for some i with $1 \leq i \leq d$. Since $\sigma^{-n}(\sigma(z)) \in \mathfrak{m} = \langle h - \lambda_i\rangle$, we have $\lambda_i = q^n\lambda_j$ for some j with $1 \leq j \leq d$. Hence $i \in \mathscr{I}$. We also have

$$\begin{aligned} J(\mathfrak{m})_m &= \bigcap\{\sigma^{\ell}(\mathfrak{m}) \mid 0 \leq \ell < n \text{ and } m \leq \ell < n+m\} \\ &= \bigcap\{h - q^{-\ell}\lambda_i \mid 0 \leq \ell < n \text{ and } m \leq \ell < n+m\} \\ &= \Big\langle \prod\{h - q^{-\ell}\lambda_i \mid 0 \leq \ell < n \text{ and } m \leq \ell < n+m\}\Big\rangle \end{aligned}$$

for $m \in \mathbb{Z}$, by Proposition 66 and the Chinese remainder theorem. Since $n = n(\mathfrak{m}) = \min\{\overline{n} > 0 \mid q^{-\overline{n}}\lambda_i \in \{\lambda_1, \ldots, \lambda_d\}\} = n(i)$, the final line is $\langle \pi_m(i)\rangle$. Thus $J(\mathfrak{m}) = P(i)$.

Conversely to the above, if $i \in \mathscr{I}$ then it is clear that $\langle h - \lambda_i\rangle \in \mathscr{M}'_{\mathrm{II}}$. Thus $\mathscr{M}'_{\mathrm{II}} = \{\langle h - \lambda_i\rangle \mid i \in \mathscr{I}\}$, and we have proven that $\operatorname{spec}(A(q,z)) = \{P(i) \mid i \in \mathscr{I}\} \cup \{0\}$. ■

3.4 Classical and Quantized $U(\mathfrak{sl}_2)$

Let $U = U(\mathfrak{sl}_2)$, the classical enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. It is the k-algebra generated by E, F, H subject to

$$[H,E] = 2E \qquad [H,F] = -2F \qquad [E,F] = H,$$

and it is also a GWA $k[H,z][F,E;\sigma : [H \mapsto H+2, z \mapsto z - H], z]$. The prime spectrum of U is known (see for example [9, Theorem 4.5]); we shall recover the result using the GWA viewpoint. By changing variables to $C = z + \frac{1}{4}H(H-2)$, get a much nicer GWA expression for U:

$$U = k[C,H][F,E;\sigma : [C \mapsto C, H \mapsto H+2], z = C - \frac{1}{4}H(H-2)].$$

Let $Z = k[C]$. For $n, m \in \mathbb{Z}$, define

$$\pi^n_m = \prod\{H - (n-1-2j) \mid 0 \leq j < n \text{ and } m \leq j < n+m\} \qquad \text{and} \qquad r_n = 4C - (n+1)(n-1).$$

For $n > 1$, define

$$P(n) := \langle \pi^n_m v_m \mid m \in \mathbb{Z}\rangle + \langle r_n\rangle \triangleleft U.$$

Theorem 135: *Assume that k has characteristic* 0*. As a set,* $\operatorname{spec}(U) = T_2 \cup T_3$, *where* $T_2 = \{\langle \mathfrak{p}\rangle \mid \mathfrak{p} \in \operatorname{spec}(Z)\}$ *and* $T_3 = \{P(n) \mid n \in \mathbb{Z}, n > 1\}$.

Proof: All ideals of U are homogeneous because $[\cdot, H]$ is a linear map that has the graded components of U as eigenspaces with distinct eigenvalues (Proposition 81).

Let $P \in \operatorname{spec}(U)$ and let $\mathfrak{p} := P \cap Z \in \operatorname{spec}(Z)$. The prime P corresponds to a prime $P' \triangleleft U/\langle\mathfrak{p}\rangle =: U' = Z'[H][F, E; \sigma, z]$, where $Z' := Z/\mathfrak{p}$. Let $\mathcal{S} = Z' \setminus \{0\}$ and let Z'' be the fraction field of Z'. Since $P' \cap Z' = 0$, the prime P' corresponds to a prime $P'' \triangleleft U'\mathcal{S}^{-1} =: U'' = Z''[H][F, E; \sigma, z]$.

By Proposition 83, all maximal ideals of $Z''[H]$ have infinite σ-orbit. Thus Corollary 71 applies and we have either $P'' = 0$, in which case $P = \langle\mathfrak{p}\rangle \in T_2$, or $P'' = J(\mathfrak{m})$ for some $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$. Suppose we are in the latter case and let $n = n(\mathfrak{m})$. From $\sigma^{-n+1}(z), \sigma(z) \in \mathfrak{m}$, we calculate that $H - (n-1), r_n \in \mathfrak{m}$. It follows that $\mathfrak{m} = \langle H - (n-1)\rangle$. Since $r_n \in Z''$ it follows that r_n vanishes in Z'', and so $\mathfrak{p} = \langle r_n\rangle$. By the Chinese remainder theorem and Proposition 66, we have $P''_m = \langle\pi^n_m\rangle$ for $m \in \mathbb{Z}$ (where π^n_m stands for the image of π^n_m in U''). Using Lemma 24 to pull back to U', we get $P'_m = \langle\pi^n_m\rangle$ for $m \in \mathbb{Z}$ (where π^n_m stands for the image of π^n_m in U'). And pulling back to A we get $P_m = \langle\pi^n_m, r_n\rangle$ for $m \in \mathbb{Z}$. Thus $P = P(n) \in T_3$. We have now shown that $\operatorname{spec}(U) \subseteq T_2 \cup T_3$.

To see that $T_2 \subseteq \operatorname{spec}(U)$, note that $U/\langle\mathfrak{p}\rangle = (Z/\mathfrak{p})[H][F, E; \sigma, z]$ is a domain for all $\mathfrak{p} \in \operatorname{spec}(Z)$.

To see that $T_3 \subseteq \operatorname{spec}(A)$, let $n \geq 1$. Now $P(n)$ is the pullback of $P' := \bigoplus_{m\in\mathbb{Z}} \langle\pi^n_m\rangle v_m \triangleleft U/\langle r_n\rangle = Z'[H][F, E; \sigma, z]$, where $Z' = Z/\langle r_n\rangle$. Let $\mathfrak{m} = \langle H - (n-1)\rangle \in \operatorname{max\,spec} Z'[H]$. In $Z'[H]$, it is easy to see by calculation that $\sigma^j(z)$ is a unit multiple of $(H - (n+1-2j))(H - (-n+1-2j))$ for $j \in \mathbb{Z}$. It follows that $n'(\mathfrak{m}) = 1$ and $n(\mathfrak{m}) = n$. From the description of $J(\mathfrak{m})$ in Proposition 66, we then see that $P' = J(\mathfrak{m})$. Thus $P(n)$ is prime. ■

We now shift to the quantized picture. Let $q \in k^\times$ be a non-root-of-unity. Let $U_q = U_q(\mathfrak{sl}_2)$ denote the standard quantization of the enveloping algebra of $\mathfrak{sl}_2$). Drawing the definition from [8, I.3], it is the k-algebra generated by E, F, K, K^{-1} subject to

$$KEK^{-1} = q^2E \qquad KFK^{-1} = q^{-2}F \qquad [E, F] = (q - q^{-1})^{-1}(K - K^{-1}).$$

It is a GWA $k[K^\pm, z][F, E; \sigma : [K \mapsto q^2K, z \mapsto z - \frac{K-K^{-1}}{q-q^{-1}}], z]$. We can express U_q more nicely by changing variables to $C := z + \frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2}$:

$$U_q = k[C, K^\pm]\left[F, E; \sigma : [K \mapsto q^2K, C \mapsto C], z = C - \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2}\right].$$

In the following theorem, we determine the prime ideals of U_q. These have already appeared in [16, Example 3.16], where the authors also take advantage of the GWA viewpoint. We will use essentially the same approach from Theorem 121. This approach is not really necessary, since (42) may be used to express U_q as a quotient of a localization of the 2×2 REA, where we already know the primes. But we leave this here as another demonstration of the machinery of section 2.6.

Let $Z = k[C]$. For $n > 0$ and $m \in \mathbb{Z}$, define

$$\pi^n_m(\pm) = \prod\left\{\left(qK \pm q^{n-2j}\right) \mid 0 \leq j < n \text{ and } m \leq j < n+m\right\}.$$

For $n > 0$ define

$$r_n(\pm) = C \pm \frac{q^n - q^{-n}}{(q - q^{-1})^2}$$

and

$$P(n, \pm) = \langle\pi^n_m(\pm)v_m \mid m \in \mathbb{Z}\rangle + \langle r_n(\pm)\rangle \triangleleft U_q.$$

Theorem 136: *Assume that q is not a root of unity. As a set,* $\operatorname{spec}(U_q) = T_2 \cup T_3$*, where* $T_2 = \{\langle\mathfrak{p}\rangle \mid \mathfrak{p} \in \operatorname{spec}(Z)\}$ *and*

$$T_3 = \{P(n, \pm) \mid n \geq 1 \text{ and } \pm \in \{+, -\}\}.$$

Proof: Since q is not a root of unity, the linear operator $a \mapsto K^{-1}aK$ has the graded components of U_q as its distinct eigenspaces. It follows that every ideal of U_q is homogeneous (Proposition 81).

Since Z is central, the contraction $P \cap Z$ of any prime P of U_q is a prime ideal of Z. Thus

$$\operatorname{spec}(U_q) = \bigcup_{\mathfrak{p} \in \operatorname{spec}(Z)} F_{\mathfrak{p}},$$

where $F_{\mathfrak{p}} := \{P \in \operatorname{spec}(U_q) \mid P \cap Z = \mathfrak{p}\} = \{P \in \operatorname{gr-spec}(U_q) \mid P \cap Z = \mathfrak{p}\}$.

Fix a $\mathfrak{p} \in \operatorname{spec}(Z)$. Define $Z' = Z/\mathfrak{p}$ and let Z'' denote the fraction field of Z'. We identify $U' := U_q/\langle \mathfrak{p} \rangle$ with $Z'[K^{\pm}][F, E; \sigma, z]$ using Proposition 12. We identify the localization of $U'' := U_q/\langle \mathfrak{p} \rangle$ at $Z' \setminus \{0\}$ with $Z''[K^{\pm}][F, E; \sigma, z]$, using Proposition 22. Now $F_{\mathfrak{p}}$ is in bijection with $\operatorname{gr-spec}(U'')$ via pullback along the quotient and localization maps $U_q \to U' \to U''$.

By Proposition 82 and the fact that q is not a root of unity, every maximal ideal of $Z''[K^{\pm}]$ has infinite σ-orbit. Thus we may apply Corollary 71 to U'' (viewing U'' as an algebra over the field Z'', so that the base ring $Z''[K^{\pm}]$ is an affine algebra). This gives $\operatorname{gr-spec}(U'') = \{0\} \cup \{J(\mathfrak{m}) \mid \mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}\}$, where $\mathscr{M}'_{\mathrm{II}} = \{\mathfrak{m} \in \max\operatorname{spec}(Z''[K^{\pm}]) \mid \sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m} \text{ for some } n > 0\}$.

Fix $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$, and let $n = n(\mathfrak{m})$. Let $P'' = J(\mathfrak{m})$. From $\sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m}$ we calculate that

$$(qK - q^n)(qK + q^n) \in \mathfrak{m}.$$

Since $\mathfrak{m}$ is maximal, it follows that $\mathfrak{m} = \langle (qK \pm q^n) \rangle$ for some $\pm \in \{+, -\}$. By the Chinese remainder theorem and Proposition 66, $P''_m = \langle \pi^n_m(\pm) \rangle$ for $m \in \mathbb{Z}$ (where we are reusing the notation $\pi^n_m(\pm)$ to refer to the image of $\pi^n_m(\pm)$ in U''). Hence, using Lemma 24, the ideals P'_m have the same expression: $P'_m = \langle \pi^n_m(\pm) \rangle$ for $m \in \mathbb{Z}$ (where we are again reusing the notation $\pi^n_m(\pm)$ to refer to the image of $\pi^n_m(\pm)$ in U'). Finally, we have $P_m = \langle \pi^n_m(\pm) \rangle + \langle \mathfrak{p} \rangle$ for $m \in \mathbb{Z}$. Now we can also calculate from $\sigma(z) \in \mathfrak{m} = \langle (qK \pm q^n) \rangle$ that $r_n(\pm)$ vanishes in $Z''[K^{\pm}]$. This implies that $r_n(\pm) \in \mathfrak{p}$, and so $\mathfrak{p} = \langle r_n(\pm) \rangle$. We conclude that $P = P(n, \pm)$.

Thus for all $\mathfrak{p} \in \operatorname{spec}(Z)$, we have $F_{\mathfrak{p}} \subseteq \{\langle \mathfrak{p} \rangle, P(n, +), P(n, -)\}$ for some $n \geq 1$. It is now clear that $\operatorname{spec}(U_q) \subseteq T_2 \cup T_3$. If $\mathfrak{p} \in \operatorname{spec}(Z)$ then $\langle \mathfrak{p} \rangle \in F_{\mathfrak{p}}$, so $T_2 \subseteq \operatorname{spec}(U_q)$.

To see that $T_3 \subseteq \operatorname{spec}(U_q)$, let $n \geq 1$ and let $\pm \in \{+, -\}$. Now $P := P(n, \pm)$ is the pullback of $P' := \bigoplus_{m \in \mathbb{Z}} \langle \pi^n_m(\pm) \rangle v_m \lhd U_q/\langle r_n(\pm) \rangle = Z'[K^{\pm}][F, E; \sigma, z]$, where $Z' := Z/\langle r_n(\pm) \rangle$. Define $\mathfrak{m} := \langle qK \pm q^n \rangle \in \max\operatorname{spec} Z'[K^{\pm}]$. In $Z'[K^{\pm}]$, it is easy to see by calculation that $\sigma^j(z)$ is a unit multiple of $(qK \pm q^{n-2j+2})(qK \pm q^{-n-2j+2})$ for $j \in \mathbb{Z}$. It follows that $n'(\mathfrak{m}) = 1$ and $n(\mathfrak{m}) = n$. From the description of $J(\mathfrak{m})$ in Proposition 66, we then see that $P' = J(\mathfrak{m})$. Thus P is prime. ■

3.5 Some Quantized Coordinate Rings

Assume that $q \in k^{\times}$ is not a root of unity. Let $A = \mathcal{O}_q(SL_2(k))$, the standard quantized coordinate ring of the Lie group $SL_2(k)$. This is defined in [8, I.1.9] as the k-algebra generated by a, b, c, d subject to

$$\begin{array}{lll} ab = qba & ac = qca & bc = cb \\ bd = qdb & cd = qdc & ad - da = (q - q^{-1})bc \\ & ad - qbc = 1. & \end{array}$$

This is a GWA

$$A = k[b, c][a, d; \sigma : [b \mapsto qb, c \mapsto qc], z = 1 + q^{-1}bc].$$

See [24, Example 7.3] for an exploration of this algebra as a GWA. Here we compute its prime spectrum using GWA theory, recovering the known result obtained using other techniques.

Theorem 137: *As a set,*

$$\operatorname{spec}(A) = \{\langle \mathfrak{p} \rangle \mid \mathfrak{p} \in \sigma\text{-}spec\,(k[b,c]) \text{ and } \mathfrak{p} \neq \langle b,c\rangle\} \cup \{\eta^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{spec}(k[a^{\pm}])\},$$

where $\eta : A \to k[a^{\pm}]$ *is given by* $b, c \mapsto 0$, $a \mapsto a$, *and* $d \mapsto a^{-1}$.

Proof: Let $D = k[b,c]$. For $\mathfrak{p} \in \sigma\text{-spec}\,(D)$, let $F_{\mathfrak{p}}$ denote the set of $P \in \operatorname{spec}(A)$ such that $(P \cap D : \sigma) = \mathfrak{p}$. By Proposition 72, we have $\operatorname{spec}(A) = \bigcup_{\mathfrak{p} \in \sigma\text{-spec}\,(D)} F_{\mathfrak{p}}$.

Consider first the case $\mathfrak{p} = \langle b,c\rangle$. It is clear that $F_{\langle b,c\rangle}$ equals the set of prime ideals of A that contain $\langle b,c\rangle$, so $F_{\langle b,c\rangle}$ is in bijection with $\operatorname{spec}(A/\langle b,c\rangle)$. We have $A/\langle b,c\rangle \cong k[a,d;\mathrm{id},1] \cong k[a^{\pm}]$ (using Propositions 12 and 5). The homomorphism η is the composite of maps $A \xrightarrow{\text{quo}} A/\langle b,c\rangle \cong k[a^{\pm}]$, so we get $F_{\langle b,c\rangle} = \{\eta^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{spec}(k[a^{\pm}])\}$.

Now fix any $\mathfrak{p} \in \sigma\text{-spec}\,(D) \setminus \{\langle b,c\rangle\}$. If we show that $F_{\mathfrak{p}} = \{\langle \mathfrak{p}\rangle\}$ then the theorem will be proven. Let $D' = D/\mathfrak{p}$ and let $A' = A/\langle \mathfrak{p}\rangle \cong D'[a,d;\sigma,z]$. Let $\mathcal{E}$ denote the set of nonzero σ-eigenvectors in D'. Let $D'' = D'\mathcal{E}^{-1}$ and let $A'' = A'\mathcal{E}^{-1} \cong D''[a,d;\sigma,z]$. Consider the $\mathbb{Z}$-grading of D by total degree of polynomials. Graded components are exactly the distinct σ-eigenspaces, so we can apply Proposition 85 to see that D'' is a σ-simple affine algebra over some field and $F_{\mathfrak{p}}$ is in correspondence with $\operatorname{spec}(A'')$. At least one of b, c must not be in $\mathfrak{p}$, and this element winds up being a unit in D''. This (and the fact that q is not a root of unity) achieves two things for us. First, conjugation by that element is a linear operator on A'' that has the graded components of A'' as distinct eigenspaces. Thus $\operatorname{spec}(A'') = \text{gr-spec}(A'')$ (Proposition 81). Second, we can apply Proposition 82 (with k possibly replaced by a different field) to conclude that all prime ideals of D'' have infinite σ-orbit. Since the $\sigma^i(z)$ are pairwise coprime in A, the same holds for their images in A''. Thus we get $\mathscr{M}'_{\mathrm{II}} = \emptyset$ in A''. Finally, Corollary 73 applies and we get $\text{gr-spec}(A'') = \{0\}$. Thus $F_{\mathfrak{p}} = \{\langle \mathfrak{p}\rangle\}$ and the theorem is proven. ∎

Let us be more explicit. Assume for the moment that k is algebraically closed. Consider the $\mathbb{Z}$-grading of $k[b,c]$ by total degree. The graded components of $k[b,c]$ are exactly the distinct σ-eigenspaces, so $\sigma\text{-spec}\,(k[b,c])$ is just the collection of graded-prime ideals of $k[b,c]$. When a $\mathbb{Z}$-grading is involved, the notions of "graded-prime" and "graded prime" coincide, so $\sigma\text{-spec}\,(k[b,c])$ consists of $\langle b,c\rangle$ (the so called irrelevant ideal), the ideals $\langle b - \lambda c\rangle$ for $\lambda \in k$ along with $\langle c\rangle$ (the closed points of one dimensional projective space) and the ideal 0 (the generic point). Now consider the homomorphism η defined in the theorem statement above. If $\mu \in k^{\times}$ then $\eta^{-1}(\langle a-\mu\rangle) = \langle b,c,a-\mu,d-\mu^{-1}\rangle$. And $\eta^{-1}(0) = \langle b,c\rangle$. Feeding all this information into the theorem above, we have found that

$$\operatorname{spec}(A) = \{\langle b-\lambda c\rangle \mid \lambda \in k\} \cup \{\langle b,c,a-\mu,d-\mu^{-1}\rangle \mid \mu \in k^{\times}\} \cup \{0, \langle c\rangle, \langle b,c\rangle\}.$$

This agrees with the result obtained in [8, Example II.2.3].

Now let $A = k[H,C][x,y;\sigma : [H \mapsto q^2 H, C \mapsto C], z = C + H^2/(q(1+q^2))]$. This GWA appears in [2] as the quantized coordinate ring $\mathcal{O}_{q^2}(\mathfrak{so}_3)$. The original construction can be found in [39, Example 4].

Theorem 138: *As a set,*

$$\operatorname{spec}(A) = \{\langle \mathfrak{p}\rangle \mid \mathfrak{p} \in \sigma\text{-}spec\,(k[H,C]) \text{ and } H \notin \mathfrak{p}\} \cup \{\eta^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{spec}(k[x,y])\},$$

where $\eta : A \to k[x,y]$ *is given by* $H \mapsto 0$, $C \mapsto yx$, $x \mapsto x$, *and* $y \mapsto y$.

Proof: Let $D = k[H,C]$. For $\mathfrak{p} \in \sigma\text{-spec}\,(D)$, let $F_{\mathfrak{p}}$ denote the set of $P \in \operatorname{spec}(A)$ such that $(P \cap D : \sigma) = \mathfrak{p}$. By Proposition 72, we have $\operatorname{spec}(A) = \bigcup_{\mathfrak{p} \in \sigma\text{-spec}\,(D)} F_{\mathfrak{p}}$.

We have $\bigcup\{F_{\mathfrak{p}} \mid \mathfrak{p} \in \sigma\text{-spec}\,(D) \text{ and } H \in \mathfrak{p}\} = \{P \in \operatorname{spec}(A) \mid H \in P\}$, which is in bijection with $\operatorname{spec}(A/\langle H\rangle)$. Now $A/\langle H\rangle \cong k[C][x,y;\mathrm{id}, z = C] \cong k[C,x,y]/\langle yx - C\rangle \cong k[x,y]$. Via these isomorphisms, the map η is exactly the quotient map $A \to A/\langle H\rangle \cong k[x,y]$. Thus

$$\bigcup\{F_{\mathfrak{p}} \mid \mathfrak{p} \in \sigma\text{-spec}\,(D) \text{ and } H \in \mathfrak{p}\} = \{\eta^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{spec}(k[x,y])\}.$$

Now fix any $\mathfrak{p} \in \sigma\text{-spec}\,(D)$ such that $H \notin \mathfrak{p}$. If we show that $F_{\mathfrak{p}} = \{\langle \mathfrak{p} \rangle\}$ then the theorem will be proven. Let $D' = D/\mathfrak{p}$ and let $A' = A/\langle \mathfrak{p} \rangle \cong D'[x, y; \sigma, z]$. Let $\mathcal{E}$ denote the set of nonzero σ-eigenvectors in D'. Let $D'' = D'\mathcal{E}^{-1}$ and let $A'' = A'\mathcal{E}^{-1} \cong D''[x, y; \sigma, z]$. Consider the $\mathbb{Z}$-grading of D in which H has degree 1 and C has degree 0. Graded components are exactly the distinct σ-eigenspaces, so we can apply Proposition 85 to see that D'' is a σ-simple affine algebra over some field and $F_{\mathfrak{p}}$ is in correspondence with $\text{spec}(A'')$. Since $H \notin \mathfrak{p}$, it becomes invertible in D''. This (and the fact that q is not a root of unity) achieves three things for us. First, conjugation by H is a linear operator on A'' that has the graded components of A'' as distinct eigenspaces. Thus $\text{spec}(A'') = \text{gr-spec}(A'')$ (Proposition 81). Second, we can apply Proposition 82 (with k possibly taken to be a different field) to conclude that all prime ideals of D'' have infinite σ-orbit. Third, the $\sigma^i(z)$ are now pairwise coprime in A'', so we get $\mathscr{M}'_{\text{II}} = \emptyset$ for A''. Finally, Corollary 73 applies and we get $\text{gr-spec}(A'') = \{0\}$. Thus $F_{\mathfrak{p}} = \{\langle \mathfrak{p} \rangle\}$ and the theorem is proven. ■

3.6 Quantized Heisenberg Algebra

Let $A = A(q, \rho) = k[t, z][x, y; \sigma : [t \mapsto q^{-1}t, z \mapsto t + \rho z], z]$, where $q, \rho \in k^{\times}$. This is the quantized Heisenberg algebra, which appears in [25, Example 6.13]. It also appears in [1], but with different parameters. It can be expressed more nicely as a GWA

$$k[H, C][x, y; \sigma : [H \mapsto q^{-1}H, C \mapsto \rho C], z = C - H/(\rho - q^{-1})]$$

by the change of variables $H = t, C = z + (\rho - q^{-1})^{-1}t$. Of course, this requires $\rho \neq q^{-1}$, but we are actually going to assume much more:

Assume that the group $\langle \rho, q \rangle \subseteq k^{\times}$ is free abelian of rank 2. This way we can be sure that the σ-eigenspaces in $k[H, C]$ are exactly the graded components with respect to the $\mathbb{Z}^2$-grading on $k[H, C]$ (where H has degree $(1, 0)$ and C has degree $(0, 1)$).

Theorem 139: *Assume that $\rho^i q^j = 1$ only if $i = j = 0$. Then*

$$\text{spec}(A) = \{\langle \mathfrak{p} \rangle \mid \mathfrak{p} \in \sigma\text{-}spec\,(D) \text{ and } H \notin \mathfrak{p}\} \cup \{\langle C, H \rangle, \langle H \rangle\}.$$

Proof: The set of prime ideals of A that contain H is in correspondence with $\text{spec}(B)$, where $B = A/\langle H \rangle = k[C][x, y; \sigma : C \mapsto \rho C, z = C]$ (Proposition 12). The set $\text{spec}(B)$ consists of $\langle C \rangle$ and the prime ideals that correspond to ones in the localization $B_c = k[C^{\pm}][x, y; \sigma : C \mapsto \rho C, z = C] \cong k[C^{\pm}][x^{\pm}; \sigma]$ (Propositions 22 and 5). The latter algebra is simple (Lemma 107 and Proposition 9, so putting all the information together we have $\{\langle C, H \rangle, \langle H \rangle\}$ as the set of prime ideals of A that contain H.

Let $D = k[H, C]$. For $\mathfrak{p} \in \sigma\text{-spec}\,(k[H, C])$, let $F_{\mathfrak{p}} = \{P \in \text{spec}(A) \mid (P \cap D : \sigma) = \mathfrak{p}\}$. The set of prime ideals of A that do not contain H can be written as $\bigcup \{F_{\mathfrak{p}} \mid \mathfrak{p} \in \sigma\text{-spec}\,(D) \text{ and } H \notin \mathfrak{p}\}$ (see Proposition 72). If we can show that $F_{\mathfrak{p}} = \{\langle \mathfrak{p} \rangle\}$ when $H \notin \mathfrak{p}$, then the theorem will be proven.

Fix $\mathfrak{p} \in \sigma\text{-spec}\,(D)$ with $H \notin \mathfrak{p}$. Let $D' = D/\mathfrak{p}$ and let $A' = A/\langle \mathfrak{p} \rangle = D'[x, y; \sigma, z]$. Let $\mathcal{E}$ denote the set of nonzero σ-eigenvectors in D'. Let $D'' = D'\mathcal{E}^{-1}$ and let $A'' = A'\mathcal{E}^{-1} = D''[x, y; \sigma, z]$. Consider the $\mathbb{Z}^2$-grading of D in which H has degree $(1, 0)$ and C has degree $(0, 1)$. Graded components are exactly the distinct σ-eigenspaces, so we can apply Proposition 85 to see that D'' is a σ-simple affine algebra over some field and $F_{\mathfrak{p}}$ is in correspondence with $\text{spec}(A'')$. Since $H \notin \mathfrak{p}$, the image of H ends up being a unit in D''. Hence all maximal ideals of D'' have infinite σ-orbit (Proposition 82, with a possibly different field in place of k). Also, conjugation by H is a linear operator on A'' that has the graded components of A'' as distinct eigenspaces, so $\text{spec}(A'') = \text{gr-spec}(A'')$ (Proposition 81). Since the $\sigma^i(z)$ are pairwise coprime in A'' (again because H is a unit), the set $\mathscr{M}'_{\text{II}}$ is empty for A''. Finally, we can apply Corollary 73 to get $\text{spec}(A'') = \text{gr-spec}(A'') = \{0\}$. It follows that $F_{\mathfrak{p}} = \{\langle \mathfrak{p} \rangle\}$. ■

3.7 Witten-Woronowicz Algebra

Let A be the GWA

$$k[H,C][x,y;\sigma:[H\mapsto r^2H, C\mapsto r^4C], z=C-\alpha],$$

where $r\in k^\times$ is not a root of unity and where

$$\alpha=\left(H-\frac{r}{1-r^2}\right)\left(H-\frac{r^3}{1-r^2}\right)\frac{1}{r^2(r+r^{-1})}.$$

This is the Witten-Woronowicz Algebra, studied in [2]. Let $D=k[H,C]$. For integers $n\geq 1$, define

$$\gamma_n=\frac{1}{r(r^2+1)}-\frac{r^2+1}{r^3(r^n+r^{-n})^2},$$
$$H_n=(r^{2n}+1)\frac{r}{1-r^4},$$

and

$$c_n=\gamma_n H^2+C.$$

For $n\geq 1$ and $m\in\mathbb{Z}$, define

$$\pi_m^n=\prod\{H-r^{-2j}H_n\mid 0\leq j<n \text{ and } m\leq j<n+m\}.$$

We will often use the same symbols to refer to the *images* of these elements of D in various quotients and localizations.

Theorem 140: *Assume that r is not a root of unity. Then*

$$\begin{aligned}\operatorname{spec}(A)=&\{\eta^{-1}(\mathfrak{p})\mid \mathfrak{p}\in\operatorname{spec}(k[x^\pm])\}\\ &\cup\{\langle\mathfrak{p}\rangle\mid \mathfrak{p}\in\sigma\text{-}spec\,(k[H,C])\setminus\{\langle H,C\rangle\}\}\\ &\cup\{\bigoplus_{m\in\mathbb{Z}}\langle\pi_m^n,c_n\rangle v_m\mid n\geq 1\},\end{aligned}$$

where $\eta: A\to k[x^\pm]$ is the homomorphism given by $H,C\mapsto 0$, $x\mapsto x$, and $y\mapsto \frac{-r^3}{(1-r^2)^2(1+r^2)}x^{-1}$.

Proof: For $\mathfrak{p}\in\sigma\text{-spec}\,(D)$, define $F_\mathfrak{p}=\{P\in\operatorname{spec}(A)\mid (P\cap D:\sigma)=\mathfrak{p}\}$. We have $\operatorname{spec}(A)=\bigcup_{\mathfrak{p}\in\sigma\text{-spec}\,(D)}F_\mathfrak{p}$, due to Proposition 72.

Consider first the case $\mathfrak{p}=\langle H,C\rangle$. Since $\langle H,C\rangle\triangleleft D$ is maximal, a prime ideal of P contains $\langle H,C\rangle$ if and only if $(P\cap D:\sigma)$ equals $\langle H,C\rangle$. It follows that $F_{\langle H,C\rangle}$ is in correspondence with $\operatorname{spec}(A/\langle H,C\rangle)$. Now we have a chain of isomorphisms

$$A/\langle H,C\rangle\cong k[x,y;\mathrm{id},z=\tfrac{-r^3}{(1-r^2)^2(1+r^2)}]\cong k[x^\pm]$$

coming from Propositions 12 and 5, and the homomorphism η is exactly the quotient map $A\to A/\langle H,C\rangle$ followed by these isomorphisms. Thus $F_{\langle H,C\rangle}=\{\eta^{-1}(\mathfrak{p})\mid\mathfrak{p}\in\operatorname{spec}(k[x^\pm])\}$.

Now consider any $\mathfrak{p}\in\sigma\text{-spec}\,(D)\setminus\{\langle H,C\rangle\}$. Make the following definitions and identifications:

$$\begin{array}{lll} D'=D/\mathfrak{p} & A'=A/\langle\mathfrak{p}\rangle=D'[x,y;\sigma,z] & \mathcal{E}=\{\text{nonzero }\sigma\text{-eigenvectors in }D'\}\\ D''=D'\mathcal{E}^{-1} & A''=A'\mathcal{E}^{-1}=D''[x,y;\sigma,z]. & \end{array}$$

We consider D to be $\mathbb{Z}$-graded such that H has degree 1 and C has degree 2. Then the σ-eigenspaces of D are exactly graded components, so we may apply Proposition 85 to see that D'' is a σ-simple affine algebra and $\operatorname{spec}(A'')$ is in correspondence with $F_\mathfrak{p}$. One of H,C must not be in $\mathfrak{p}$, and whichever it is becomes a unit in D''. Conjugation by that element defines a linear operator on A'' that has the $\mathbb{Z}$-graded components of A'' as its eigenspaces. It follows (Proposition 81) that $\operatorname{spec}(A'')=\operatorname{gr-spec}(A'')$. One also has, due to whichever of H,C became a unit in D'', that every maximal ideal of D'' has infinite σ-orbit (Proposition 82).

Claim: Suppose that $n \geq 1$ and $c_n \in \mathfrak{p}$. Then $\mathfrak{p} = \langle c_n \rangle$, no other $c_{n'} \in \mathfrak{p}$ for $n' \geq 1$, and $\langle H - H_n \rangle_{D''} \in \mathscr{M}'_{\mathrm{II}}$.
Proof: Since $c_n \in \mathfrak{p}$, the ideal $\mathfrak{p}$ corresponds to an ideal of $D/\langle c_n \rangle$, which is isomorphic (as a $\mathbb{Z}$-graded algebra) to $k[H]$ (with its usual grading). The only nonzero graded-prime ideal of $k[H]$ is $\langle H \rangle$, which pulls back to $\langle H, C \rangle$ along $D \xrightarrow{\text{quo}} D/\langle c_n \rangle \cong k[H]$. Since $\mathfrak{p} \neq \langle H, C \rangle$, it follows that $\mathfrak{p} = \langle c_n \rangle$.

Now suppose that also $c_{n'} \in \mathfrak{p}$, where $n' \geq 1$. Then $c_n, c_{n'} \in \mathfrak{p}$ gives $(\gamma_n - \gamma_{n'})H^2 \in \mathfrak{p}$. We cannot have $H^2 \in \mathfrak{p}$ since $\mathfrak{p}$ is σ-prime and $H \in \mathfrak{p}$ would lead to the contradiction $\mathfrak{p} = \langle H, C \rangle$. Thus $\gamma_n = \gamma_{n'}$. This implies that $(r^n + r^{-n})^2 = (r^{n'} + r^{-n'})^2$. Expanding and rearranging, we get

$$r^{2n} - r^{2n'} = r^{-2n'} - r^{-2n} = r^{-2(n+n')}(r^{2n} - r^{2n'}).$$

If $n' \neq n$, this gives $r^{-2(n+n')} = 1$, which is a contradiction since $n, n' \geq 1$. Thus $n' = n$.

Note that $H_n \neq 0$. Again using the graded-isomorphism $D' \cong k[H]$, we see that $\langle H - H_n \rangle_{D'}$ is a maximal ideal of D' that contains no nonzero homogeneous polynomial. Thus $\mathfrak{m} := \langle H - H_n \rangle_{D''}$ is a maximal ideal of D''. By a calculation (that can be found in Appendix E), one can show that the polynomials $\sigma(z), \sigma^{-n+1}(z)$ vanish at the point given by $H = H_n$ and $C = \gamma_n H_n^2$. Considering that $c_n = 0$ in D'', this means we have $\sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m}$. Thus $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$.

Claim: Suppose that $\mathfrak{m} \in \mathscr{M}'_{\mathrm{II}}$ and let $n = n(\mathfrak{m})$. Then $\mathfrak{m} = \langle H - H_n \rangle_{D''}$ and $\mathfrak{p} = \langle c_n \rangle$.
Proof: In D, a calculation (that can be found in Appendix E) shows that the ideal generated by $\sigma(z)$ and $\sigma^{-n+1}(z)$ contains $H - H_n$ and c_n. So it follows from $\sigma(z), \sigma^{-n+1}(z) \in \mathfrak{m}$ that $H - H_n, c_n \in \mathfrak{m}$. It follows that $c_n \in \mathfrak{p}$, for c_n is a σ-eigenvector and otherwise would have become a unit in D''. Using the previous claim we may now conclude that $\mathfrak{p} = \langle c_n \rangle$ and $\mathfrak{m} = \langle H - H_n \rangle_{D''}$.

From the two claims above we deduce the following: If $\mathfrak{p}$ contains no c_n for $n \geq 1$, then $\mathscr{M}'_{\mathrm{II}} = \emptyset$. On the other hand if $c_n \in \mathfrak{p}$ for some $n \geq 1$, then $\mathscr{M}'_{\mathrm{II}} = \{\langle H - H_n \rangle\}_{D''}$, we have $\mathfrak{p} = \langle c_n \rangle$, and we have $n(\langle H - H_n \rangle_{D''}) = n$. So if $\mathfrak{p}$ contains no c_n for $n \geq 1$ then we may apply Corollary 73 to get gr-spec$(A'') = \{0\}$, from which it follows that $F_{\mathfrak{p}} = \{\langle \mathfrak{p} \rangle\}$. Suppose now that $c_n \in \mathfrak{p}$ for some $n \geq 1$. Since $\mathfrak{p} = \langle c_n \rangle$, there is an isomorphism $D' \cong k[H]$ that fixes H and preserves the $\mathbb{Z}$-grading (where $k[H]$ has its usual grading). Identifying D' with $k[H]$ using this isomorphism, we see that $D'' = k[H^{\pm}]$, and so A'' is a domain and $0 \in$ gr-spec(A''). By Corollary 73, we now have gr-spec$(A'') = \{0, J(\langle H - H_n \rangle_{D''})\}$. Observe that the $H - r^{-2j}H_n$ are pairwise coprime in D'' as $j \in \mathbb{Z}$ varies. Using the Chinese remainder theorem and Proposition 66, we see that $J(\langle H - H_n \rangle_{D''})\} = \bigoplus_{m \in \mathbb{Z}} \langle \pi_m^n \rangle_{D''} v_m$. Using Lemma 24 to pull back $J(\langle H - H_n \rangle_{D''})$ along the localization $A' \to A''$, and then pulling back along the quotient $A \to A'$, we get $F_{\mathfrak{p}} = \left\{\mathfrak{p}, \bigoplus_{m \in \mathbb{Z}} \langle \pi_m^n, c_n \rangle v_m \right\}$.

The theorem follows from our descriptions of all the $F_{\mathfrak{p}}$. ■

A Localization

There are a few aspects of noncommutative localization that make an appearance throughout this work and that rely on the noetherian hypothesis. For the reader's convenience, we lay them out here. Proposition 141 says that localization "commutes" with factoring out an ideal, and it is a standard fact. Theorem 142 says that the usual correspondence of prime ideals along a localization is a homeomorphism, also a standard fact. Finally, Lemma 143 provides a way to describe the pullback of a prime ideal along a localization by using a "nice" generating set.

Proposition 141: *Let $\mathcal{S}$ be a right denominator set in a right noetherian ring R. Let I be an ideal of*

R, with extension I^e to $R\mathcal{S}^{-1}$. Then:

- *I^e is an ideal of $R\mathcal{S}^{-1}$.*
- *$\bar{\mathcal{S}} := \{s + I \mid s \in \mathcal{S}\}$ is a denominator set of R/I.*
- *The canonical homomorphism $\phi : R/I \to (R\mathcal{S}^{-1})/I^e$ gives a right ring of fractions for R/I with respect to $\bar{\mathcal{S}}$. That is, there is an isomorphism $\bar{\phi} : (R/I)\bar{\mathcal{S}}^{-1} \cong (R\mathcal{S}^{-1})/I^e$ making the following diagram commute:*

$$\begin{tikzcd} R \arrow[r, "loc"] \arrow[d, two heads] & R\mathcal{S}^{-1} \arrow[r, two heads, "quo"] & R\mathcal{S}^{-1}/I^e \\ R/I \arrow[urr, "\exists!\phi"] \arrow[rr, "loc"'] & & (R/I)\bar{\mathcal{S}}^{-1} \arrow[u, "\bar{\phi}", "\cong"'] \end{tikzcd}$$

Proof: By [21, Theorem 10.18a], I extends to an ideal of $R\mathcal{S}^{-1}$. It is clear that $\bar{\mathcal{S}}$ is a right Ore set of R/I. It is automatically right reversible due to the noetherian hypothesis; see [21, Proposition 10.7]. We will use the universal property that characterizes rings of fractions (e.g. see [21, Proposition 10.4]). The homomorphism ϕ is uniquely defined because I is in the kernel of the upper row of the diagram. Since ϕ maps $\bar{\mathcal{S}}$ to a collection of units, $\bar{\phi}$ is uniquely defined such that the diagram commutes. It is surjective because $rs^{-1} + I^e$ is the image of $(r + I)(s + I)^{-1}$ for $r \in R$ and $s \in \mathcal{S}$. For injectivity, suppose that $rs^{-1} \in I^e$. Then $r1^{-1} \in I^e$, so $r \in I^{ec}$, the contraction of I^e to R. By [21, Theorem 10.15b], this implies that $rs' \in I$ for some $s' \in \mathcal{S}$. That is, $0 = (r + I)1^{-1} \in (R/I)\bar{\mathcal{S}}^{-1}$. ■

Theorem 142: *Let $\mathcal{S}$ be a right denominator set in a right noetherian ring R. Then contraction and extension of prime ideals are inverse homeomorphisms:*

$$\operatorname{spec}(R\mathcal{S}^{-1}) \;\approx\; \{Q \in \operatorname{spec}(R) \mid Q \cap \mathcal{S} = \emptyset\}. \tag{100}$$

Proof: The bijection of sets in (100) is given to us by [21, Theorem 10.20], so we only need to show that closed sets are preserved. The closed subsets of $\{Q \in \operatorname{spec}(R) \mid Q \cap \mathcal{S} = \emptyset\}$ are

$$V_{\mathcal{S}}(J) := \{Q \in \operatorname{spec}(R) \mid Q \supseteq J \text{ and } Q \cap \mathcal{S} = \emptyset\},$$

for ideals J of R. Observe that (100) preserves finite unions and inclusions of prime ideals. Since R is right noetherian, every ideal of R has finitely many minimal primes over it, and the same goes for $R\mathcal{S}^{-1}$. It follows that the topological spaces in (100) are noetherian, so every closed set is a finite union of irreducible closed sets. Hence it suffices to show that (100) preserves irreducible closed sets. Irreducible closed sets of $\operatorname{spec}(R\mathcal{S}^{-1})$ are of the form $V(P)$ for primes $P \triangleleft R\mathcal{S}^{-1}$. Irreducible closed sets of $\{Q \in \operatorname{spec}(R) \mid Q \cap \mathcal{S} = \emptyset\}$ are of the form $V_{\mathcal{S}}(P)$ for primes $P \triangleleft R$ such that $P \cap \mathcal{S} = \emptyset$. Since (100) preserves inclusions of prime ideals, it sends any $V_{\mathcal{S}}(P)$ to $V(P\mathcal{S}^{-1})$. ■

Lemma 143: *Let R be a right noetherian ring, $\mathcal{S} \subseteq R$ a right denominator set, and $\phi : R \to R\mathcal{S}^{-1}$ the localization map. Let $\mathcal{G} \subseteq R$ and assume the following:*

1. *The* right *ideal P generated by $\mathcal{G}$ is a two-sided ideal of R.*
2. *Either P is a prime ideal of R disjoint from $\mathcal{S}$, or $\langle\phi(\mathcal{G})\rangle$ is a prime ideal of $R\mathcal{S}^{-1}$ and $(R/P)_R$ is $\mathcal{S}$-torsionfree.*
3. *For all $g \in \mathcal{G}$ and $s \in \mathcal{S}$,*
$$g\mathcal{S} \cap sP \neq \emptyset.$$

Then

$$P = \phi^{-1}(\langle\phi(\mathcal{G})\rangle).$$

That is, the ideal of $R\mathcal{S}^{-1}$ generated by $\phi(\mathcal{G})$ contracts to the ideal of R generated by $\mathcal{G}$.

Proof: Assumption 3 guarantees that the right ideal of $R\mathcal{S}^{-1}$ generated by $\phi(\mathcal{G})$ is a two-sided ideal. Let superscripts "e" and "c" denote extension and contraction of ideals along ϕ. Observe that

$$\begin{aligned}\langle\phi(\mathcal{G})\rangle &= \{\sum_{i=1}^{n}\phi(g_i)\phi(r_i)\phi(s_i)^{-1} \mid n\in\mathbb{Z}_{\geq 0},\ r_i\in R,\ s_i\in\mathcal{S},\ g_i\in\mathcal{G} \text{ for } 1\leq i\leq n\}\\ &= \{\sum_{i=1}^{n}\phi(g_ir_i)\phi(s)^{-1} \mid s\in\mathcal{S},\ n\in\mathbb{Z}_{\geq 0},\ r_i\in R,\ g_i\in\mathcal{G} \text{ for } 1\leq i\leq n\} \qquad (101)\\ &= \{\phi(a)\phi(s)^{-1} \mid s\in\mathcal{S},\ a\in\langle\mathcal{G}\rangle\} = P^e.\end{aligned}$$

In line (101), we used the fact that it is possible to get a "common right denominator" for a finite list of right fractions; see [21, Lemma 10.2]. Now assumption 2 implies that P is a prime ideal of R disjoint from $\mathcal{S}$, either trivially or by [21, Theorem 10.18b]. To finish, we use the correspondence between prime ideals disjoint from $\mathcal{S}$ and prime ideals of $R\mathcal{S}^{-1}$:

$$P = P^{ec} = \langle\phi(\mathcal{G})\rangle^c = \phi^{-1}(\langle\phi(\mathcal{G})\rangle).$$ ■

Note that assumption 3 of Lemma 143 holds trivially whenever $\mathcal{G}$ or $\mathcal{S}$ is central.

B Chinese Remainder Theorem

This appendix serves to clarify the way in which the Chinese Remainder Theorem (CRT) gets applied in section 3.1.4.1.

Let R be a commutative ring with pairwise comaximal ideals $I_1,\ldots,I_n$, and let $\Pi := I_1\cdots I_n$ denote their product. The CRT says that the homomorphism

$$R/\Pi \to R/I_1\times\cdots\times R/I_n, \qquad (102)$$

whose components are induced by canonical projections, is an isomorphism. This implies that there is a bijective correspondence between ideals J of R that contain Π, and tuples $(J_1,\ldots,J_n)$ of ideals of R such that $J_i\supseteq I_i$ for all i. Let us describe the correspondence explicitly.

There are $e_1,\ldots e_n\in R$ such that $e_i\equiv 1 \bmod I_i$ and $e_i\equiv 0 \bmod I_j$ for $j\neq i$. These are just the pairwise orthogonal idempotents (when taken modulo Π) corresponding to the ring decomposition in (102); they also satisfy

$$e_1+\cdots+e_n\equiv 1 \mod \Pi.$$

The ring $(R/\Pi)(e_i+\Pi)$ is the copy of the ring R/I_i that is (non-unitally) contained in R/Π via (102). Explicitly, the correspondence of elements is given by

$$(re_i+\Pi) \ \leftrightarrow\ (r+I_i). \qquad (103)$$

Let J be an ideal of R that contains Π. Projecting J/Π to the copy $(R/\Pi)(e_i+\Pi)$ of R/I_i yields $(J/\Pi)(e_i+\Pi)$, which corresponds to $(J+I_i)/I_i \vartriangleleft R/I_i$ via (103). Going back in the other direction, suppose that $J_1,\ldots,J_n$ are ideals of R such that $J_i\supseteq I_i$ for all i. Then J_i/I_i is an ideal of $R/I_1\times\cdots\times R/I_n$, and it corresponds to $(J_ie_i+\Pi)/\Pi \vartriangleleft R/\Pi$ via (103). The sum of these over i is the ideal of R/Π corresponding to the tuple $(J_1,\ldots,J_n)$. Thus we have the following explicit description of how ideals are carried across (102):

Proposition 144: *Let R be a commutative ring with pairwise comaximal ideals $I_1,\ldots,I_n$, and let $\Pi :=$ $I_1\cdots I_n$ denote their product. Let $e_1,\ldots e_n\in R$ be such that $e_i\equiv 1 \bmod I_i$ and $e_i\equiv 0 \bmod I_j$ for $j\neq i$.*

There is a bijective correspondence between ideals J of R that contain Π, and tuples $(J_1, \ldots, J_n)$ of ideals of R such that $J_i \supseteq I_i$ for all i, and it is given by:

$$\begin{array}{ccc} J & \mapsto & (J + I_1, \ldots, J + I_n) \\ J_1 e_1 + \cdots + J_n e_n + \Pi & \leftarrow\!\shortmid & (J_1, \ldots, J_n). \end{array}$$

C Derivations and Poisson Brackets

This appendix contains tools that are used to build Poisson GWAs from the ground up, in section 2.7.

Definition 145: Given a k-algebra R and a bimodule ${}_R M_R$, we denote the upper triangular matrix algebra

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in R,\ b \in M \right\}$$

by $\left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix} \right]$. Given a function $\delta : R \to M$, we define $\delta^{\mathbf{H}} : R \to \left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix} \right]$ to be the function

$$r \to \begin{pmatrix} r & \delta(r) \\ 0 & r \end{pmatrix}.$$

Given a function $\phi : R \to \left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix} \right]$, we define $\phi^{\mathbf{D}} : R \to M$ to be $\pi_{12} \circ \phi$, where $\pi_{12} : \left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix} \right] \to M$ is the projection. Observe that $\delta^{\mathbf{H}^{\mathbf{D}}} = \delta$, and that $\phi^{\mathbf{D}^{\mathbf{H}}} = \phi$ if $\phi(r)$ has the form $\left[\begin{smallmatrix} r & \cdot \\ 0 & r \end{smallmatrix} \right]$ for all $r \in R$.

This notation has been introduced to make it convenient to apply the following standard trick for constructing derivations:

Proposition 146: *Let R be a k-algebra and let ${}_R M_R$ be a bimodule. A function $\delta : R \to M$ is a k-derivation if and only if $\delta^{\mathbf{H}} : R \to \left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix} \right]$ is an algebra homomorphism.*

Proof: It is obvious that δ is k-linear if and only if $\delta^{\mathbf{H}}$ is k-linear. Observe that for $a, b \in R$ one has

$$\begin{aligned} \delta^{\mathbf{H}}(a)\delta^{\mathbf{H}}(b) &= \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix} \begin{pmatrix} b & \delta(b) \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & a\delta(b) + \delta(a)b \\ 0 & ab \end{pmatrix}, \\ \delta^{\mathbf{H}}(ab) &= \begin{pmatrix} ab & \delta(ab) \\ 0 & ab \end{pmatrix}. \end{aligned}$$

Thus δ has the Leibniz property if and only if $\delta^{\mathbf{H}}$ is multiplicative. ■

Proposition 147: *A derivation (from a k-algebra R to an R-R-bimodule) is determined by the values it takes on a generating set for the algebra.*

Proof: This is an easy consequence of Proposition 146, since algebra homomorphisms are determined by their values on generators. ■

Throughout this appendix, the notation $R[x]$ denotes a polynomial ring.

Proposition 148: *Let R be a commutative k-algebra and M an R-module. Let $\delta : R \to M$ be a derivation and let x' be any element of M. There is a unique derivation $\delta_1 : R[x] \to M$ extending δ and sending $x \mapsto x'$.*

Proof: Define $\phi_1 : R[x] \to \left[\begin{smallmatrix} R[x] & M \\ 0 & R[x] \end{smallmatrix} \right]$ to be the extension of

$$R \xrightarrow{\delta^{\mathbf{H}}} \left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix} \right] \hookrightarrow \left[\begin{smallmatrix} R[x] & M \\ 0 & R[x] \end{smallmatrix} \right] \tag{104}$$

to $R[x]$ that sends x to

$$\begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}. \tag{105}$$

This is a well-defined algebra homomorphism because (104) is a homomorphism by Proposition 146, and because (105) commutes with matrices of the form

$$\begin{pmatrix} r & \delta(r) \\ 0 & r \end{pmatrix}.$$

The desired δ_1 is then simply $\phi_1{}^{\mathbf{D}}$, a derivation by Proposition 146. The uniqueness of δ_1 is guaranteed by Proposition 147. ∎

Proposition 149: *Let R be a commutative k-algebra and M an R-module. Let $\delta : R \to M$ be a derivation and I be an ideal of R. If $\delta(I) \subseteq IM$, then δ induces a derivation $R/I \to M/IM$*

Proof: Consider the algebra homomorphism $q : \left[\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right] \to \left[\begin{smallmatrix} R/I & M/IM \\ 0 & R/I \end{smallmatrix}\right]$ that acts as the standard quotient map in each coordinate. Define $\phi_1 : R \to \left[\begin{smallmatrix} R/I & M/IM \\ 0 & R/I \end{smallmatrix}\right]$ to be the homomorphism $q \circ \delta^{\mathbf{H}}$. If $\delta(I) \subseteq IM$, then ϕ_1 annihilates I and therefore induces a homomorphism $\phi_2 : R/I \to \left[\begin{smallmatrix} R/I & M/IM \\ 0 & R/I \end{smallmatrix}\right]$. The desired derivation would then be $\phi_2{}^{\mathbf{D}}$. ∎

Definition 150: Given a k-algebra R and a bimodule ${}_RM_R$, we denote by $\mathrm{Der}_k(R, M)$ the vector space of k-derivations $R \to M$. A *k-biderivation* from R to M is a k-bilinear map $R \times R \to M$ which is a derivation in each argument. We shall refer to k-derivations and k-biderivations simply as derivations and biderivations.

When R is commutative, $\mathrm{Der}_k(R, M)$ becomes an R-module in an obvious way. Biderivations can then be expressed as iterated derivations:

Proposition 151: *Let R be a commutative k-algebra and let M be an R-module. A biderivation from R to M is equivalent to a derivation from R to the R-module $\mathrm{Der}_k(R, M)$. That is, a given function $\{-,-\} : R \times R \to M$ is a biderivation if and only if the mapping $a \mapsto (b \mapsto \{a, b\})$ is an element of $\mathrm{Der}_k(R, \mathrm{Der}_k(R, M))$.*

Proof: Let $\{-,-\}$ be any function $R \times R \to M$ and let Δ denote the mapping $a \mapsto (b \mapsto \{a, b\})$, a function from R to $\mathrm{Hom}_{\mathbf{Set}}(R, M)$, the set of functions $R \to M$. The set $\mathrm{Hom}_{\mathbf{Set}}(R, M)$ is an R-module in an obvious way.

The function $\{-,-\}$ is k-bilinear if and only if Δ is an element of $\mathrm{Hom}_k(R, \mathrm{Hom}_k(R, M))$ (using the well known natural isomorphism $\mathrm{Hom}_k(R \otimes_k R, M) \cong \mathrm{Hom}_k(R, \mathrm{Hom}_k(R, M))$). For $\{-,-\}$ to satisfy the Leibniz condition in its right argument, the necessary and sufficient condition is that

$$\Delta(a)(bc) = \{a, bc\} = c\,\{a, b\} + b\,\{a, c\} = c\Delta(a)(b) + b\Delta(a)(c)$$

for all $a, b, c \in R$. In other words, one needs exactly that $\Delta(a)$ is derivation for all $a \in R$. For $\{-,-\}$ to satisfy the Leibniz condition in its left argument, the necessary and sufficient condition is that

$$\Delta(ab)(c) = \{ab, c\} = a\,\{b, c\} + b\,\{a, c\} = a\Delta(b)(c) + b\Delta(a)(c) = (a\Delta(b) + b\Delta(a))(c)$$

for all $a, b, c \in R$. In other words, one needs exactly that $\Delta(ab) = a\Delta(b) + b\Delta(a)$ for all $a, b \in R$. Thus if $\{-,-\}$ is k-bilinear, then it is a biderivation if and only if $\Delta \in \mathrm{Der}_k(R, \mathrm{Der}_k(R, M))$. ∎

Proposition 152: *A biderivation (from a k-algebra R to an R-R-bimodule) is determined by the values it takes on ordered pairs of elements from a generating set.*

Proof: Using Proposition 151 to express biderivations in terms of derivations, apply Proposition 147. ■

One consequence is that for a homomorphism to preserve Poisson brackets it is enough for it to do so on algebra generators:

Proposition 153: *Let R, S be commutative Poisson k-algebras and let $\phi : R \to S$ be an algebra homomorphism. Let $\mathcal{G}$ be an algebra generating set for R. If $\phi(\{a, b\}) = \{\phi(a), \phi(b)\}$ for all $a, b \in \mathcal{G}$, then ϕ is a Poisson homomorphism.*

Proof: View S as an R-module via ϕ. To say that ϕ preserves Poisson brackets is to say that the biderivations $\{\phi(-), \phi(-)\}$ and $\phi(\{-,-\})$ from R to S coincide. The result follows from Proposition 152. ■

Proposition 154: *Let R be a commutative k-algebra with a biderivation $\{-,-\}_R$ to an R-module M. Let $\delta^l_x, \delta^r_x : R \to M$ be derivations and let ξ be an element of M. There is a unique biderivation $\{-,-\}$ from $R[x]$ to M extending $\{-,-\}_R$ such that $\{x, -\}|_R = \delta^l_x$, $\{-, x\}|_R = \delta^r_x$, and $\{x, x\} = \xi$.*

Proof: Let $\Delta_0 \in \mathrm{Der}_k(R, \mathrm{Der}_k(R, M))$ be the derivation given by $a \mapsto (b \mapsto \{a, b\}_R)$, as in Proposition 151. Define a function $\Delta_1 : R \to \mathrm{Der}_k(R[x], M)$ as follows: for $a \in R$, let $\Delta_1(a)$ be the unique extension of $\Delta_0(a)$ to $R[x]$ that sends x to $\delta^r_x(a)$, which exists by Proposition 148. Let us check that Δ_1 is a derivation: for $a, b, c \in R$ and $\beta \in k$ we have

$$\Delta_1(\beta a + b)(c) = \Delta_0(\beta a + b)(c) = \beta\Delta_0(a)(c) + \Delta_0(b)(c) = \beta\Delta_1(a)(c) + \Delta_1(b)(c) = (\beta\Delta_1(a) + \Delta_1(b))(c)$$
$$\Delta_1(\beta a + b)(x) = \delta^r_x(\beta a + b) = \beta\delta^r_x(a) + \delta^r_x(b) = \beta\Delta_1(a)(x) + \Delta_1(b)(x) = (\beta\Delta_1(a) + \Delta_1(b))(x),$$

so Δ_1 is k-linear. For $a, b, c \in R$ we have

$$\Delta_1(ab)(c) = \Delta_0(ab)(c) = a\Delta_0(b)(c) + b\Delta_0(a)(c) = a\Delta_1(b)(c) + b\Delta_1(a)(c) = (a\Delta_1(b) + b\Delta_1(a))(c)$$
$$\Delta_1(ab)(x) = \delta^r_x(ab) = a\delta^r_x(b) + b\delta^r_x(a) = a\Delta_1(b)(x) + b\Delta_1(a)(x) = (a\Delta_1(b) + b\Delta_1(a))(x),$$

so Δ_1 is indeed a derivation. Again using Proposition 148, define the derivation $\widehat{\delta^l_x} : R[x] \to M$ to be the extension of δ^l_x to $R[x]$ that sends x to ξ. Finally, use Proposition 148 for a third time to define $\Delta_2 \in \mathrm{Der}_k(R[x], \mathrm{Der}_k(R[x], M))$ to be the extension of Δ_1 that sends x to $\widehat{\delta^l_x}$. The desired $\{-,-\}$ is $\Delta_2(-)(-)$, which is a biderivation by Proposition 151. The uniqueness of this biderivation is given by Proposition 152. ■

Proposition 155: *Let R be a commutative k-algebra with a biderivation $\{-,-\}$ to an R-module M. Let I be an ideal of R such that $\{I, R\} + \{R, I\} \subseteq IM$. Then $\{-,-\}$ induces a biderivation from the quotient R/I to the quotient M/IM.*

Proof: Define $\Delta_1 : R \to \mathrm{Der}_k(R/I, M/IM)$ by letting $\Delta_1(r)$ be, for each $r \in R$, the derivation from R/I to M/IM induced by $\{r, -\}$ via Proposition 149; this works because $\{R, I\} \subseteq IM$. It is easily checked that Δ_1 is a derivation. Since $\{I, R\} \subseteq IM$, we have $\Delta_1(I) = 0$. Hence Proposition 149 applies again to give an induced derivation $\Delta_2 : R/I \to \mathrm{Der}_k(R/I, M/IM)$ (note that $I\,\mathrm{Der}_k(R/I, M/IM) = 0$). The derivation Δ_2 corresponds, via Proposition 151, to the desired biderivation from R/I to M/IM. ■

Proposition 156: *Let R be a commutative k-algebra with generating set $\mathcal{G}$, and let $\{-,-\}$ be a biderivation of R. Assume that*

$$\{a, b\} = -\{b, a\} \quad \textit{and}$$
$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$$

for all $a, b, c \in \mathcal{G}$. Then $\{-,-\}$ is a Poisson bracket on R.

Proof: For any $a \in R$ let $S_a = \{b \in R \mid \{a,b\} = -\{b,a\}\}$. Since $\{-,-\}$ is a biderivation, it is easy to check that S_a is a linear subspace of R which contains 1. It is also closed under multiplication; assuming that $b, b' \in S_a$ we have:

$$\{a, bb'\} = \{a,b\}\, b' + \{a,b'\}\, b = -(\{b,a\}\, b' + \{b',a\}\, b) = -\{bb',a\}\,.$$

Thus S_a is a subalgebra of R for any $a \in R$. By our assumption we then have that $S_g = R$ for $g \in \mathcal{G}$. That is, $\{g,b\} = -\{b,g\}$ for all $b \in R$ and $g \in \mathcal{G}$. Fixing any $b \in R$ and moving the minus sign, we can express this by saying that $\{b,g\} = -\{g,b\}$ for all $g \in \mathcal{G}$. It follows that $S_b = R$. Since this holds for arbitrary $b \in R$, we have shown that $\{-,-\}$ is antisymmetric.

For any $a, b \in R$, let $S_{a,b} = \{c \in R \mid \{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0\}$. Since $\{-,-\}$ is a biderivation, it is easy to check that $S_{a,b}$ is a linear subspace of R which contains 1. It is also closed under multiplication. To show this, it helps to use the notation $\delta_a = \{a,-\}$ and the fact that a commutator of derivations is a derivation. Assuming that $c, c' \in S_a$ we have:

$$\begin{aligned}
\{a,\{b,cc'\}\} + \{b,\{cc',a\}\} + \{cc',\{a,b\}\} &= [\delta_a,\delta_b](cc') + \{cc',\{a,b\}\} \\
&= c'[\delta_a,\delta_b](c) + c[\delta_a,\delta_b](c') + c'\,\{c,\{a,b\}\} + c\,\{c',\{a,b\}\} \\
&= -c'\,\{c,\{a,b\}\} - c\,\{c',\{a,b\}\} + c'\,\{c,\{a,b\}\} + c\,\{c',\{a,b\}\} \\
&= 0.
\end{aligned}$$

Thus $S_{a,b}$ is a subalgebra of R for any $a, b \in R$. By our assumption we then have that $S_{g,g'} = R$ for $g, g \in \mathcal{G}$. That is, $a \in S_{g,g'}$ for all $a \in R$ and $g, g' \in \mathcal{G}$ Using the symmetry of the Jacobi identity, this implies that $g' \in S_{g,a}$ for all $a \in R$ and $g, g' \in \mathcal{G}$. It follows that $S_{g,a} = R$ for all $a \in R$ and $g \in \mathcal{G}$. That is, $b \in S_{g,a}$ for all $a, b \in R$ and $g \in \mathcal{G}$. Again using the symmetry of the Jacobi identity, this means that $g \in S_{a,b}$ for all $a, b \in R$ and $g \in \mathcal{G}$. It follows that $S_{a,b} = R$ for all $a, b \in R$. Thus we have shown that $\{-,-\}$ satisfies the Jacobi identity. ■

Proposition 157: *Poisson derivations preserve Poisson centers.*

Proof: Let R be a commutative Poisson k-algebra with $\alpha : R \to R$ a Poisson derivation. If $z \in R$ is Poisson central, then for any $r \in R$ we have:

$$\{\alpha(z), r\} = \alpha(\{z,r\}) - \{z, \alpha(r)\} = 0.$$ ■

D The Dixmier-Moeglin Equivalence and Localization

We show here two useful facts: First, it is often automatic for GWAs to satisfy one direction of the Dixmier-Moeglin equivalence. Second, the full Dixmier-Moeglin equivalence for an algebra is often preserved by localization.

Definition 158: A noetherian k-algebra A *satisfies the nullstellensatz over* k if the following two conditions hold. First, A must be a *Jacobson ring*– this means that all its prime factor rings have zero Jacobson radical. Second, for every simple left A-module, its division ring of endomorphisms must be algebraic over k.

The following definitions appear in [32, 1.6.10 and 9.1.4].

Definition 159: Let $R \subseteq S$ be k-algebras. If S is generated over R by $x_1, \ldots, x_n$, each x_i satisfies $Rx_i + R = x_iR + R$, and each $[x_i, x_j]$ is in $\sum_{\ell=1}^{n} x_\ell R + R$, then S is called an *almost normalizing* *extension* of R. If S can be obtained from R by a finite number of extensions, each being either a finite module extension or an almost normalizing extension, then S is said to be *constructible from* R. If S is constructible from k, then S is simply called a *constructible* k-algebra. For example, any affine commutative algebra is constructible.

Remark 160: Definition 158 appears in [32, 9.1.4] and in [8, II.7.14] with a slight difference, but in the noetherian setting there is no difference due to [32, Lemma 9.1.2]. To avoid worrying about the difference, we will only deal with the nullstellensatz in a noetherian setting.

Proposition 161: *If a noetherian k-algebra R is constructible, then any GWA $R[x, y; \sigma, z]$ satisfies the nullstellensatz over k.*

Proof: It is easy to see that any GWA $W = R[x, y; \sigma, z]$ is an almost normalizing extension of R and hence is constructible from R. Since we assumed R is constructible from k, we have that W is constructible from k. Note that W is noetherian since R is (Proposition 4), so we need not worry about the point made in Remark 160. By [32, Theorem 9.4.21], we have that W satisfies the nullstellensatz over k. ■

Proposition 162: *Let A be a noetherian k-algebra satisfying the Dixmier-Moeglin equivalence, and let $\mathcal{S} \subseteq A$ be a right denominator set consisting of regular elements. Assume that the localization $A\mathcal{S}^{-1}$ satisfies the nullstellensatz over k. Then $A\mathcal{S}^{-1}$ satisfies the Dixmier-Moeglin equivalence and its primitive ideals correspond, via extension and contraction, to the primitives of A that are disjoint from $\mathcal{S}$.*

Proof: Let $P \in \operatorname{spec}(A)$ with $P \cap \mathcal{S} = \emptyset$, and let $Q \in \operatorname{spec}(A\mathcal{S}^{-1})$ be its extension.

> **Claim:** If P is locally closed, then Q is locally closed.
> **Proof:** From Theorem 142, contraction of primes gives a continuous injection $\operatorname{spec}(A\mathcal{S}^{-1}) \to \operatorname{spec}(A)$. The preimage of a locally closed set under a continuous map is locally closed. Since Q maps to P, it follows that Q is locally closed when P is.
>
> **Claim:** P is rational if and only if Q is rational
> **Proof:** Let $\overline{S}$ denote the image of S in A/P. The claim follows from the fact that $Z(\mathrm{Fract}(A/P)) = Z(\mathrm{Fract}((A/P)\overline{S}^{-1})) = Z(\mathrm{Fract}((AS^{-1})/Q))$.

Consider the following implications:

$$\begin{array}{ccccc}
P \text{ locally closed} & \Longleftrightarrow & P \text{ primitive} & \Longleftrightarrow & P \text{ rational} \\
\Downarrow & & & & \Updownarrow \\
Q \text{ locally closed} & \Longrightarrow & Q \text{ primitive} & \Longrightarrow & Q \text{ rational}
\end{array}$$

The top row is the Dixmier-Moeglin equivalence for A. The bottom row follows from the fact that $A\mathcal{S}^{-1}$ satisfies the nullstellensatz over k (see [8, II.7.15]). The left and right sides are the claims above. The Dixmier-Moeglin equivalence for $A\mathcal{S}^{-1}$ follows, and P is primitive if and only if Q is primitive. ■

E Calculations

This appendix includes some mathematica code to help verify certain statements in section 3.7.

In[43]:= $$\alpha[\mathrm{H\_}] = \left(\mathrm{H} - \frac{\mathrm{r}}{1 - \mathrm{r}^2}\right)\left(\mathrm{H} - \frac{\mathrm{r}^3}{1 - \mathrm{r}^2}\right)\frac{1}{\mathrm{r}^2\left(\mathrm{r} + \mathrm{r}^{-1}\right)};$$

z[H_, C_] := C − α[H];

(∗This is σ^{j}(z) evaluated at a given H and C∗)

sigmajz[j_, H_, C_] = z$\left[\mathrm{r}^{2\,\mathrm{j}}\,\mathrm{H},\ \mathrm{r}^{4\,\mathrm{j}}\,\mathrm{C}\right]$;

$$\gamma[\mathrm{n\_}] = \frac{1}{\mathrm{r}\left(\mathrm{r}^2 + 1\right)} - \frac{\mathrm{r}^2 + 1}{\mathrm{r}^3\,(\mathrm{r}^{\mathrm{n}} + \mathrm{r}^{-\mathrm{n}})^2};$$

$$\mathrm{H}[\mathrm{n\_}] = \left(\mathrm{r}^{2\,\mathrm{n}} + 1\right)\frac{\mathrm{r}}{1 - \mathrm{r}^4};$$

In[48]:= (∗ If the following evaluates to 0, then σ(z) is in the maximal ideal $<\mathrm{H}-\mathrm{H}_{\mathrm{n}}>$ of D" ∗)

Simplify[sigmajz[1, H[n], γ[n] H[n]^2]]

(∗ If the following evaluates to 0, then $\sigma^{-\mathrm{n}+1}$(z) is in the maximal ideal $<\mathrm{H}-\mathrm{H}_{\mathrm{n}}>$ of D" ∗)

Simplify[sigmajz[−n + 1, H[n], γ[n] H[n]^2]]

Out[48]= 0

Out[49]= 0

In[50]:= (∗ The following calculation verifies that $\mathrm{H}-\mathrm{H}_{\mathrm{n}}$ is in the ideal $<\sigma(\mathrm{z}),\sigma^{-\mathrm{n}+1}(\mathrm{z})>$ of D ∗)

$$\mathrm{Simplify}\left[\frac{\mathrm{r}^2\left(-1 + \mathrm{r}^2\right)^2\left(1 + \mathrm{r}^2\right)}{\left(-1 + \mathrm{r}^4\right)\left(-1 + \mathrm{r}^{2\,\mathrm{n}}\right)}\left(\mathrm{r}^{-4}\ \mathrm{sigmajz}[1,\ \mathrm{H},\ \mathrm{C}] - \mathrm{r}^{4\,\mathrm{n}-4}\ \mathrm{sigmajz}[-\mathrm{n} + 1,\ \mathrm{H},\ \mathrm{C}]\right) == \mathrm{H} - \mathrm{H}[\mathrm{n}]\right]$$

Out[50]= True

In[51]:= (∗ Using the fact that $\mathrm{H}-\mathrm{H}_{\mathrm{n}}$ is in $<\sigma(\mathrm{z}),\sigma^{-\mathrm{n}+1}(\mathrm{z})>$, we find that $\gamma_{\mathrm{n}}\mathrm{H}^2-\mathrm{C}$ is also in $<\sigma(\mathrm{z}),\sigma^{-\mathrm{n}+1}(\mathrm{z})>$ ∗)

$$\mathrm{Simplify}\left[(\mathrm{H} - \mathrm{H}[\mathrm{n}])\left((\mathrm{H} + \mathrm{H}[\mathrm{n}])\,\mathrm{H}[\mathrm{n}]^{-2}\left(\frac{\mathrm{r}^2 - \mathrm{r}^{2\,\mathrm{n}} - \mathrm{r}^{4+2\,\mathrm{n}} + \mathrm{r}^{2+4\,\mathrm{n}}}{\mathrm{r}\left(-1 + \mathrm{r}^2\right)^2\left(1 + \mathrm{r}^2\right)^3}\right)\right) - \mathrm{r}^{-4}\ \mathrm{sigmajz}[1,\ \mathrm{H}[\mathrm{n}],\ \mathrm{C}] == \gamma[\mathrm{n}]\,\mathrm{H}^2 - \mathrm{C}\right]$$

Out[51]= True

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